SKP Engineering College

Tiruvannamalai – 606611

A Course Material

on

Transform and Partial Differential Equations

By

K. Srinivasan

S. Dhanasekar

K. Suresh

M. Sathiyamoorthy

Assistant Professor

Mathematics Department

Quality Certificate

This is to Certify that the Electronic Study Material

Subject Code: MA6351

Subject Name: Transforms and Partial Differential Equations

Year/Sem: II / III

Being prepared by us and it meets the knowledge requirement of the University curriculum.

Signature of the Author

Name: K. Srinivasan

S. Dhanasekar

K. Suresh

M. Sathiyamoorthy

Designation: Assistant Professor

This is to certify that the course material being prepared by Mr. K. Srinivasan , Mr. S.Dhanasekar, Mr. K. Suresh and Mr. M. Sathiyamoorthy is of the adequate quality. They have referred more than five books and one among them is from abroad author.

Signature of HD

Name: S. Gokulakrishnan

Seal:

Signature of the Principal

Name: Dr.V.Subramania Bharathi

Seal:

MA6351 TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

SYLLABUS

OBJECTIVES:

- To introduce Fourier series analysis which is central to many applications in engineering apart from its use in solving boundary value problems.
- To acquaint the student with Fourier transform techniques used in wide variety of situations.
- To introduce the effective mathematical tools for the solutions of partial differential equations that model several physical processes and to develop Z transform techniques for discrete time systems.

UNIT I PARTIAL DIFFERENTIAL EQUATIONS

Formation of partial differential equations – Singular integrals -- Solutions of standard types of first order partial differential equations - Lagrange's linear equation -- Linear partial differential equations of second and higher order with constant coefficients of both homogeneous and non-homogeneous types.

UNIT II FOURIER SERIES

Dirichlet's conditions – General Fourier series – Odd and even functions – Half range sine series – Half range cosine series – Complex form of Fourier series – Parseval's identity – Harmonic analysis.

UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Classification of PDE – Method of separation of variables - Solutions of one dimensional wave equation – One dimensional equation of heat conduction – Steady state solution of two dimensional equation of heat conduction (excluding insulated edges).

UNIT IV FOURIER TRANSFORMS

Statement of Fourier integral theorem – Fourier transform pair – Fourier sine and cosine transforms – Properties – Transforms of simple functions – Convolution theorem – Parseval's identity.

UNIT V Z - TRANSFORMS AND DIFFERENCE EQUATIONS

Z- Transforms - Elementary properties – Inverse Z - transform (using partial fraction and residues) – Convolution theorem - Formation of difference equations – Solution of difference equations using Z - transform.

OUTCOMES:

- The understanding of the mathematical principles on transforms and partial differentialequations would provide them the ability to formulate and solve some of the physical problems of engineering.

TEXT BOOKS:

1. Veerarajan T., "Transforms and Partial Differential Equations", Tata McGraw Hill Education Pvt. Ltd., New Delhi, Second reprint, 2012.

2. Grewal B.S., "Higher Engineering Mathematics", 42nd Edition, Khanna Publishers, Delhi, 2012.

3. Narayanan S., Manicavachagom Pillay.T.K and Ramanaiah.G "Advanced Mathematics for Engineering Students" Vol. II & III, S.Viswanathan Publishers Pvt Ltd. 1998.

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1. Bali. N.P and Manish Goyal, "A Textbook of Engineering Mathematics", 7th Edition, Laxmi Publications Pvt Ltd, 2007.

2. Ramana. B.V., "Higher Engineering Mathematics", Tata McGraw Hill Publishing Company Limited, New Delhi, 2008.

3. Glyn James, "Advanced Modern Engineering Mathematics", 3rd Edition, Pearson Education, 2007.

4. Erwin Kreyszig, "Advanced Engineering Mathematics", 8 th Edition, Wiley India, 2007.

5. Ray Wylie C and Barrett.L.C, "Advanced Engineering Mathematics" Tata McGraw Hill Education Pvt Ltd, Sixth Edition, New Delhi, 2012.

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CONTENTS

UNIT I PARTIAL DIFFERENTIAL EQUATIONS

PART – A

Problem 1 Form the partial differential equation by eliminating a and b from $z=(x^2+a^2)(y^2+b^2)$. **[CO 1 - H2]**

Solution:

$$z=(x^2+a^2)(y^2+b^2) \quad -(1)$$

Differentiating (1) partially w.r.t x and y we get

$$p=\frac{\partial z}{\partial x}=(y^2+b^2)2x \quad -(2)$$

$$q=\frac{\partial z}{\partial y}=(x^2+a^2)2y \quad -(3)$$

Multiplying Eqn. (2) and Eqn (3) $\Rightarrow$

$$pq=(x^2+a^2)(y^2+b^2)4xy$$

$pq=4xyz$ [using (1)]

Problem 2 Obtain the partial differential equation by eliminating arbitrary constants a and b from $(x-a)^2+(y-b)^2+z^2=1$. **[CO 1 - H2 - Nov/Dec 2015]**

Solution:

$$(x-a)^2+(y-b)^2+z^2=1 \quad -(1)$$

Differentiating (1) partially w.r.t x and y we get

$$2(x-a)+2zp=0$$

$$\Rightarrow x-a=-zp \quad -(2)$$

$$2(y-b)+2zq=0$$

$$\Rightarrow y-b=-zq \quad -(3)$$

Substituting (2) & (3) in (1) we get

$$z^2p^2+z^2q^2+z^2=1$$

i.e., $z^2(p^2+q^2+1)=1$

Problem 3 From the partial differential equation by eliminating f from $f\left(x^2+y^2+z^2, x+y+z\right)=0$. **[CO 1 - H2 - May/Jun 2016]**

Solution:

We know that if f (u, v) = 0

then u = f (v)

$$\therefore x^2+y^2+z^2=f(x+y+z) \qquad -(1)$$

Differentiating (1) partially w.r. t x and y

We get

$$2x+2zp=f'(x+y+z)(1+p) \qquad -(2)$$

$$2y+2zq=f'(x+y+z)(1+q)$$

Divide (2) & (3)

$$\frac{x+zp}{y+zq}=\frac{1+p}{1+q}$$

$$x+qx+zp+zpq=y+py+zq+zpq$$

$$(z-y)p+(x-z)q=y-x$$

Problem 4 Form the partial differential equation by eliminating f from $z=x^2+2f\left(\frac{1}{y}+\log x\right)$. **[CO 1 - H2]**

Solution:

Let $z=x^2+2f\left(\frac{1}{y}+\log x\right) \qquad -(1)$

Differentiate (1) w.r. t x and y

$$\frac{\partial z}{\partial x}=p=2x+2f'\left(\frac{1}{y}+\log x\right)\left(\frac{1}{x}\right) \qquad -(2)$$

$$\frac{\partial z}{\partial y}=q=2f'\left(\frac{1}{y}+\log x\right)\left(\frac{-1}{y^2}\right) \qquad -(3)$$

Eliminating f' from (2) & (3)

$$\therefore \frac{p-2x}{q}=\frac{-1}{x}\left(y^2\right)$$

$$px-2x^2=-qy^2$$

$$px+qy^2=2x^2$$

Problem 5 Obtain the partial differential equation by eliminating the arbitrary constants a & b from $z = xy + y\sqrt{x^2 - a^2} + b$. **[CO 1 - H2]**

Solution:

$$z = xy + y\sqrt{x^2 - a^2} + b \qquad - (1)$$

Differentiating (1) partially w.r. t x and y

$$p = \frac{\partial z}{\partial x} = y + y\left[\frac{1}{2\sqrt{x^2 - a^2}}\right][2x]$$

$$p = y + \frac{yx}{\sqrt{x^2 - a^2}}$$

$$\frac{p}{y} = 1 + \frac{x}{\sqrt{x^2 - a^2}}$$

$$\frac{p}{y} - 1 = \frac{x}{\sqrt{x^2 - a^2}} \qquad - (2)$$

$$q = \frac{\partial z}{\partial y} = x + \sqrt{x^2 - a^2}$$

$$q - x = \sqrt{x^2 - a^2} \qquad - (3)$$

Multiplying (2) & (3)

$$\left(\frac{p}{y} - 1\right)(q - x) = \frac{x}{\sqrt{x^2 - a^2}}\sqrt{x^2 - a^2}$$

$$\left(\frac{p}{y} - 1\right)(q - x) = x$$

$$(p - y)(q - x) = xy$$

$$pq - xp - yq + xy = xy$$

$$px + qy = pq$$

Problem 6 Find the complete integral of $p + q = pq$ where $p = \frac{\partial z}{\partial x}$ *and* $q = \frac{\partial z}{\partial y}$. **[CO 1 - H2]**

Solution:

$$p + q = pq \qquad - (1)$$

This is of the form $f(p, q) = 0$

Let $z = ax + by + c$ - (2) be the complete solution of the partial differential equation.

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

(1) reduces to $a+b=ab$

$$a=b(a-1)$$

$$b=\frac{a}{a-1}$$

$$\therefore z=ax+\left(\frac{a}{a-1}\right)y+c$$

Problem 7 Obtain the complete integral of $z=px+qy+p^2+q^2$.

Solution: **[CO 1 - H2 - Nov/Dec 2015]**

$z=px+qy+p^2+q^2$ - (1)

This equation is of the form $z=px+qy+f(p,q)$ (clairaut's type)

$\therefore$ the complete integral is $z=ax+by+a^2+b^2$.

Problem 8 Solve $p(1+q)=qz$. **[CO 1 - L1]**

Solution:

$p(1+q)=qz$ - (1)

This equation is of the form $f(z,p,q)=0$

$z=f(x+ay)$ be the solution

$x+ay=u$ $z=f(u)$

$p=\frac{dz}{du}$ $q=\frac{adz}{du}$

(1) reduces to

$$\frac{dz}{du}\left(1+a\frac{dz}{du}\right)=a\frac{dz}{du}z$$

$$1+a\frac{dz}{du}=az$$

$$a\frac{dz}{du}=az-1$$

$$\frac{dz}{du}=z-\frac{1}{a}$$

$$\frac{dz}{z-\frac{1}{a}}=du$$

Integrating $\log\left(z-\frac{1}{a}\right)=u+b$

i.e., $\log\left(z-\frac{1}{a}\right)=x+ay+b$ is the complete solution.

Problem 9 Solve the equation $p\tan x+q\tan y=\tan z$. **[CO 1 - L1]**

Solution:

Given $p\tan x+q\tan y=\tan z$

This equation is of the form $P_p+Q_q=R$

When P = tan x Q = tan y R = tan z

The subsidiary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

i.e., $$\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z}$$

Considering the first two,

$$\frac{dx}{\tan x}=\frac{dy}{\tan y}$$

Sign,

$$\int\cot x dx=\int\cot y dy$$

$$\log\sin x=\log\sin y+\log a$$

$$\log\left(\frac{\sin x}{\sin y}\right)=\log a$$

$$\frac{\sin x}{\sin y} = a$$

$$i.e., a = \frac{\sin x}{\sin y}$$

Take,

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\int \frac{dy}{\tan y} = \int \frac{dz}{\tan z}$$

$$\int \cot y dy = \int \cot z dz$$

$$\log \sin y = \log \sin z + \log b$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log b$$

$$\therefore b = \frac{\sin y}{\sin z}$$

Hence the general solution is $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

Problem 10 Solve $\left(D^3 - 3DD'^2 + 2D'^3\right)Z = 0$. **[CO 1 - L1]**

Solution:

Substituting D = m, & $D^1 = 1$

The Auxiliary equation is $m^3 - 3m + 2 = 0$

m = 1, 1, -2

Complimentary function is $\phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x)$

i.e., $Z = \phi_1(y+1) + x\phi_2(y+x) + \phi_3(y-2x)$

Problem 11 Find the general solution of $4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$. **[CO 1 - H2]**

Solution:

$$\left(4D^2 - 12DD^1 + 9D^{12}\right)Z = 0$$

The auxiliary equation is $4m^2 - 12m + 9 = 0$

$4m^2 - 6m - 6m + 9 = 0$

$2m(2m - 3) - 3(2m - 3) = 0$

$$(2m-3)^2 = 0$$

$$m = \frac{3}{2} \text{ (twice)}$$

$$\text{C.F.} = \phi_1\left(y+\frac{3}{2}x\right)+x\phi_2\left(y+\frac{3}{2}x\right)$$

$\therefore$ the General solution is Z = C.F. + P.I

$$z = \phi_1\left(y+\frac{3}{2}x\right)+x\phi_2\left(y+\frac{3}{2}x\right)$$

Problem 12 Solve $\left(D^3 + DD'^2 - D^2D' - D'^3\right)z = 0$. **[CO 1 - L1]**

Solution:

The Auxiliary equation is

$$m^3 - m^2 + m - 1 = 0$$

$$m^2(m-1)+(m-1) = 0$$

$$\left(m^2+1\right)(m-1) = 0$$

$$i.e.\ m = 1, i, -i$$

$\therefore$ The general solution is $Z = \phi_1(y+x)+\phi_2(y+ix)+\phi_3(y-ix)$

Problem 13 Find the particular integral of $\left(D^3 + D^2D' - DD'^2 - D'^3\right)z = e^x \cos 2y$.

Solution: **[CO 1 - H2]**

$$\text{P.I.} = \frac{1}{D^3+D^2D'-DD'^2-D'^3}e^x\cos 2y$$

$$= e^x \frac{\cos 2y}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3}$$

$$= e^x \text{ R.P. of } \frac{e^{2iy}}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3}$$

$$= e^x \text{ R.P. of } \frac{e^{2iy}}{1+2i+4+8i}$$

$$= \frac{e^x}{5} \text{ R.P. of } \frac{1-2i}{(1+2i)(1-2i)}(\cos 2y + i\sin 2y)$$

$$= \frac{1}{5}e^x.\frac{1}{5}(\cos 2y + 2\sin 2y)$$

$$\text{P.I.} = \frac{e^x}{25}(\cos 2y + 2\sin 2y)$$

Problem 14 Solve $\left(D^2 - DD' + D' - 1\right)z = 0$. **[CO 1 - L1]**

Solution:

$$\left(D^2 - DD' + D' - 1\right) = 0$$

$$(D-1)(D-D'+1) = 0$$

This is of the form

$$\left(D - m_1 D^1 - \alpha_1\right)\left(D - m_2 D^1 - \alpha_2\right) = 0$$

$$m_1 = 0, \alpha_1 = 1, m_2 = 1, \alpha_2 = -1$$

C.F. is $Z = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$

$$Z = e^{x} f_1(y) + e^{-x} f_2(y + x)$$

Problem 15 Solve $(D - D' - 1)(D - D' - 2)z = e^{2x-y}$. **[CO 1 - H2 - Apr/May 2014]**

Solution:

This is of the form

$$(D - m_1 D' - C_1)(D - m_2 D' - C_2)....(D - m_n D' - C_n)Z = 0$$

Hence $m_1 = 1 \quad c_1 = 1 \quad m_2 = 1 \quad c_2 = 1 \; c_2 = 2$

Hence the C.F. is $z = e^{x}\phi_1(y + x) + e^{2x}\phi_2(y + x)$

$$\text{P.I.} \quad = \frac{e^{2x-y}}{(D - D' - 1)(D - D' - 2)}$$

$$= \frac{e^{2x-y}}{(2+1-1)(2+1-2)}$$

$$= \frac{1}{2} e^{2x-y}$$

Hence, the complete solution is $Z = e^{x}\phi_1(y + x) + e^{2x}\phi_2(y + x) + \frac{1}{2} e^{2x-y}$

PART – B

Problem 16

a. From the partial differential equation by eliminating f and ϕ from $z = f(y) + \phi(x+y+z)$. **[CO 1 - H2]**

Solution:

$$z = f(y) + \phi(x+y+z) \qquad -(1)$$

Differentiating partially with respect to x and y, we get

$$P = \phi'(x+y+z)(1+p) \qquad -(2)$$

$$q = f'(y) + \phi'(x+y+z)(1+q) \qquad -(3)$$

$$r = \phi'(x+y+z).r + \phi''(x+y+z)(1+p)^2 \qquad -(4)$$

$$s = \phi'(x+y+z).s + \phi''(x+y+z)(1+p)(1+q) \qquad -(5)$$

$$t = f''(y) + \phi'(x+y+z)t + \phi''(x+y+z)(1+q)^2 \qquad -(6)$$

From (4) $r\{1-\phi'(x+y+z)\} = (1+p)^2 \phi''(x+y+z) \qquad -(7)$

From (5) $s\{1-\phi'(x+y+z)\} = (1+p)(1+q)\phi''(x+y+z) \qquad -(8)$

Dividing (7) & (8) we get

$$\frac{r}{s} = \frac{(1+p)}{(1+q)}$$

$$(1+q)r = (1+p)s$$

b. Solve $(D^2 - 2DD')z = e^{2x} + x^3 y$. **[CO 1 - H2 - Nov/Dec 2014]**

Solution:

Auxiliary Equation is given by $m^2 - 2m = 0$

i.e., m (m – 2) =0

m = 0, m = 2

C.F. $= f_1(y) + f_2(y+2x)$

$$PI_1 = \frac{1}{(D^2 - 2DD')} e^{2x}$$

$$= \frac{e^{2x}}{4} \text{ (Replace D by 2 and D}' \text{ by 0)}$$

$$PI_2 = \frac{1}{(D^2 - 2DD')} x^3 y$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1} x^3 y$$

$$= \frac{1}{D^2}\left(1+\frac{2D'}{D}+......\right)x^3y$$

$$= \frac{1}{D^2}\left(x^3y+\frac{2x^3}{D}\right)$$

$$= \frac{x^5}{20}y+2.\frac{x^6}{4.5.6}$$

$$= \frac{x^5}{20}y+\frac{x^6}{60}$$

The complete solution is

$$Z = C.F. + PI_1 + PI_2$$

$$Z = f_1(y)+f_2(y+2x)+\frac{e^{2x}}{4}+\frac{x^5y}{20}+\frac{x^6}{20}$$

Problem 17

a. Find the complete integral of $p+q=x+y$. **[CO 1 - H2]**

Solution:

The given equation does not contain z explicitly and is variable separable.
That is the equation can be rewritten as p – x = y – q =a, say - (!)

$\therefore p = a + x$ and $q = y - a$

Now $dz = pdx + qdy$

$$dz = (a+x)dx + (y-a)dy \qquad (2)$$

Integrating both sides with respect to he concerned variables, we get

$$z = \frac{(a+x)^2}{2}+\frac{(y-a)^2}{2}+b \qquad - (3)$$

when a and b are arbitrary constants.

b. Solve $y^2p - xyq = x(z-2y)$. **[CO 1 - H2]**

Solution:

This is a Lagrange's linear equation of the form $Pp+Qq=R$

The subsidiary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)}$$

Taking I & II ratios

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$xdx = -ydy$$

Integrating,

$$\frac{x^2}{2} = \frac{-y^2}{2} + C$$

$$x^2 + y^2 = 2c = c_1$$

Taking II & III ratios $\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$$(Z-2y)dy = -ydz$$

$$ydz + zdy = 2ydy$$

$$d(yz) = 2ydy$$

Integrating,

$$yz = y^2 + c_2$$

$$yz - y^2 = c_2$$

the solution is $\phi\left(x^2 + y^2, yz - y^2\right) = 0$

Problem 18

a. Find the singular integral of the partial differential equation $z = px + qy + p^2 - q^2$. **[CO 1 - H2 - Apr/May 2014]**

Solution:

The given equation $z = px + qy + p^2 - q^2$ is a clairaut's type equation.

Hence the complete solution is $z = ax + by + a^2 - b^2$ -(1)

<u>To get singular Solution :</u>

Differentiate (1) w.r.t. a and b

$0 = x + 2a$ - (2)

$0 = y - 2b$ - (3)

$\therefore a = \frac{-x}{2}$ and $b = \frac{y}{2}$

Substituting in (1),

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2 - y^2}{4}$$

$$z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4}$$

$4z = y^2 - x^2$ is the singular solution.

b. Solve $\left(D^2+4DD'-5D'^2\right)z=3e^{2x-y}+\sin(x-2y)$. **[CO 1 - H2]**

Solution:

The Auxiliary equation,

$$m^2+4m-5=0$$
$$m^2+5m-m-5=0$$
$$m(m+5)-(m+5)=0$$
$$(m-1)(m+5)=0$$
$$m=1,-5$$

$\therefore$ C.F. $=f(y+x)+g(y-5x)$

$$PI_1 = \frac{1}{D^2+4DD'-5D'^2}3e^{2x-y}$$
$$=\frac{3e^{2x-y}}{4-8-5}$$
$$=\frac{3}{-9}e^{2x-y}$$
$$=\frac{-1}{3}e^{2x-y}$$

$$PI_2 = \frac{1}{D^2+4DD'-5D'^2}\sin(x-2y)$$
$$=\frac{1}{-1+4(2)-5(-4)}\sin(x-2y)$$
$$=\frac{1}{-1+8+20}\sin(x-2y)$$
$$=\frac{1}{27}\sin(x-2y)$$

$\therefore$ the complete solution is $Z=CF+PI_1+PI_2$

i.e. $Z=f(y+x)+g(y-5x)-\frac{1}{3}e^{2x-y}+\frac{1}{27}\sin(x-2y)$

Problem 19 a. Find the general solution of $(3z-4y)p+(4x-2z)q=2y-3x$.

Solution: **[CO 1 - H2]**

This is a Lagrange's linear equation of the form $Pp+Qq=R$.

The subsidiary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{3z-4y}=\frac{dy}{4x-2z}=\frac{dz}{2y-3x} \qquad -(1)$$

Use Lagrangian multipliers x, y, z we get each ratio is

$$\frac{xdx+ydy+zdz}{3xz-4xy+4xy-2yz+2yz-3xz}=\frac{\frac{1}{2}d\left(x^2+y^2+z^2\right)}{0}$$

Hence $d\left(x^2+y^2+z^2\right)=0$

Integrating both the sides,

$\therefore x^2+y^2+z^2=C_1 \qquad -(2)$

using multipliers 2, 3, 4 each of equation (1) is

$$=\frac{2dx+3dy+4dz}{6z-8y+12x-6z+8y-12x}$$

$$=\frac{2dx+3dy+4dz}{0}$$

$\therefore 2dx+3dy+4dz=0$

Hence $2x+3y+4z=C_2 \qquad -(3)$

$\therefore$ the General solution is $\phi\left(x^2+y^2+z^2, 2x+3y+4z\right)=0$

b. Solve $\left(D^2-2DD'+D'^2-3D+3D'+2\right)z=\left(e^{3x}+2e^{-2y}\right)^2$. **[CO 1 - L3]**

Solution:

$$\left(D^2-2DD'+D'^2-3D+3D'+2\right)z=\left[(D-D'-2)(D-D'-1)\right]z$$

$$\left[(D-D'-2)(D-D'-1)\right]z=e^{6x}+4e^{3x-2y}+4e^{-4y}$$

$$m_1=1,\alpha_1=2,m_2=1,\alpha_2=1$$

$$\therefore \text{C.F.}=e^{2x}f(y+x)+e^{x}f(y+x)$$

$$PI_1 = \frac{1}{(D-D'-2)(D-D'-1)}e^{6x}$$

$$= \frac{1}{(6-0-2)(6-0-1)} e^{6x}$$

$$= \frac{1}{20} e^{6x}$$

$$PI_2 = \frac{1}{(D-D'-2)(D-D'-1)} 4e^{3x-2y}$$

$$= \frac{4}{(3+2-2)(3+2-1)} e^{3x-2y}$$

$$= \frac{4}{3(4)} e^{3x-2y}$$

$$= \frac{1}{3} e^{3x-2y}$$

$$PI_3 = \frac{1}{(D-D'-2)(D-D'-1)} 4e^{-4y}$$

$$= \frac{4e^{-4y}}{(0+4-2)(0+4-1)}$$

$$= \frac{4}{(2)(3)} e^{-4y}$$

$$= \frac{2}{3} e^{-4y}$$

$\therefore$ the general solution

$$Z = e^{2x} f(y+x) + e^{x} f(y+x) + \frac{1}{20} e^{6x} + \frac{1}{3} e^{3x-2y} + \frac{2}{3} e^{-4y}$$

Problem 20

a. Form the differential equation by eliminating the arbitrary function f and g in $z = f(x+y)g(x-y)$. **[CO 1 - H2]**

Solution:

$$z = f(x+y)g(x-y)$$

Let $u = x+y$ $\quad v = x-y$

$$Z = f(u).g(v) \qquad -(1)$$

Differentiating partially with respect to x and y, we get

$$p = f(u).g'(v) + f'(u).g(v) \qquad -(2)$$

$$q = f(u)g'(v)(-1) + f'(u)g(v) \qquad -(3)$$

$$r = f(u)g''(v) + 2f'(u)g'(v) + f''(u).g(v) \qquad -(4)$$

$$s = f(u)g''(v)(-1) + f''(u).g(v) \qquad -(5)$$

$$t = f(u)g''(v) - 2f'(u)g'(v) + f''(u)g(v) \qquad -(6)$$

Subtracting (4) from (6) , we get

$$r-t=4f'(u).g'(v) \qquad - (7)$$

From (2) & (3), we get

$$p^2-q^2=4f(u).g(v).f'(u).g'(v)$$

$$= Z\ (r-t) \text{ from (1) \& (7)}$$

i.e., $z\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right)=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2$

b. Solve the equation $(pq-p-q)(z-px-qy)=pq$. **[CO 1 - H1]**

Solution:

Rewriting the given equation as $Z=px+qy+\frac{pq}{pq-p-q}$ - (1)

we identify it as a clairaut's type equation. Hence its complete solution is

$$Z=ax+by+\frac{ab}{ab-a-b} \qquad - (2)$$

The general solution of (1) is found out as usual from (2).

Let us now find the singular solution of (1).

Differentiating (2) partially with respect to a and then b, we get

$$0=x+\frac{(ab-a-b)b-ab(b-1)}{(ab-a-b)^2}$$

i.e., $0=x-\frac{b^2}{(ab-a-b)^2}$ - (3)

and similarly

$$0=y-\frac{a^2}{(ab-a-b)^2} \qquad - (4)$$

From (3) & (4), we get $\frac{a^2}{b^2}=\frac{y}{x}$ or $\frac{a}{\sqrt{y}}=\frac{b}{\sqrt{x}}=k$, say

$\therefore a=k\sqrt{y}$ and $b=k\sqrt{x}$

Using there values in (3) we get

$$k^2x-\left(k^2\sqrt{xy}-k\sqrt{y}-k\sqrt{x}\right)^2x=0$$

$$k^2x=\left(k^2\sqrt{xy}-k\sqrt{y}-k\sqrt{x}\right)x$$

i.e., $\left(k\sqrt{xy}-\sqrt{x}-\sqrt{y}\right)=1$

$$k=\frac{1+\sqrt{x}+\sqrt{y}}{\sqrt{xy}}$$

Hence $a=\frac{1+\sqrt{x}+\sqrt{y}}{\sqrt{x}}$ and $b=\frac{1+\sqrt{x}+\sqrt{y}}{\sqrt{y}}$

Also

$$\frac{ab}{ab-a-b}=\frac{1}{1-\frac{1}{b}-\frac{1}{a}}=\frac{1}{1-\frac{\sqrt{y}}{1+\sqrt{x}+\sqrt{y}}-\frac{\sqrt{x}}{1+\sqrt{x}+\sqrt{y}}}$$

$$=1+\sqrt{x}+\sqrt{y}$$

Using these values in (2), the singular solution of (1) is

$$z=\sqrt{x}\left(1+\sqrt{x}+\sqrt{y}\right)+\sqrt{y}\left(1+\sqrt{x}+\sqrt{y}\right)+\left(1+\sqrt{x}+\sqrt{y}\right)$$

$$z=\left(1+\sqrt{x}+\sqrt{y}\right)^2$$

Problem 21

a. Form the partial differential equation by eliminating f from $f\left(x^2+y^2+z^2, z^2-2xy\right)=0$. **[CO 1 - H2 - Apr/May 2014]**

Solution:

$$f\left(x^2+y^2+z^2, z^2-2xy\right)=0$$

Let $u=x^2+y^2+z^2$ and $v=z^2-2xy$ then

the given equation is $f(u,v)=0$ - (1)

Differentiating (1) partially w.r.t. x and y respectively

we get $\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial x}=0$ and $\frac{\partial f}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}.\frac{\partial v}{\partial y}=0$

$$\Rightarrow (2x+2zp)\frac{\partial f}{\partial u}+(2zp-2y)\frac{\partial f}{\partial v}=0 \qquad - (4)$$

$$(2y+2zq)\frac{\partial f}{\partial u}+(2zq-2x)\frac{\partial f}{\partial v}=0 \qquad -(5)$$

from (4) and (5)

we get $\begin{vmatrix} x+zp & zp-y \\ y+zq & zq-x \end{vmatrix}=0$

$$(x+zp)(zq-x)=(y+zq)(zp-y)$$

$$zqx-x^2+z^2pq-xzp=zpy-y^2+z^2pq-zqy$$

$$-zp(x+y)+zq(x+y)=x^2-y^2$$

$$-zp(x+y)+zq(x+y)=(x-y)(x+y)$$

$$-zp+zq=x-y$$

$$zp-zq=y-x$$

$$z(p-q)=y-x$$

b. Solve the equation $p\left(1-q^2\right)=q\left(1-z\right)$. **[CO 1 - H1]**

Solution:

The given equation is of the from $f\left(z,p,q\right)=0$

$P\left(1-q^2\right)=q\left(p-z\right)$ - (1)

Let $z=f\left(x+ay\right)$ be the solution of (1)

If $x+ay=u$ then $z=f\left(u\right)$

If $p=\frac{dz}{du}$ and $q=\frac{adz}{du}$

(1) reduces as

$$\frac{\partial z}{\partial u}\left[1-a^2\left(\frac{dz}{du}\right)^2\right]=\frac{adz}{du}(1-z)$$

i.e., $\frac{dz}{du}\left[1-a^2\left(\frac{dz}{du}\right)^2-a+az\right]=0$

As z is not a constant, $\frac{dz}{du}\neq 0$

$\therefore 1-a^2\left(\frac{dz}{du}\right)^2-a+az=0$

i.e., $a^2\left(\frac{dz}{du}\right)^2=az+1-a$

$a\frac{dz}{du}=\sqrt{az+1-a}$ - (3)

solving (3), we get

$$a\int\frac{dz}{\sqrt{az+1-a}}=\int du$$

$$2\sqrt{az+1-a}=u+b$$

$$2\sqrt{az+1-a}=x+ay+b$$

$$4\left(az+1-a\right)=\left(x+ay+b\right)^2$$

Problem 22

a. Solve the equation $p^2+q^2=z^2\left(x^2+y^2\right)$. **[CO 1 - H1]**

Solution:

The given equation dues not belong to any of the standard types.

It can be rewritten as $\left(z^{-1}p\right)^2+\left(z^{-1}q\right)^2=x^2+y^2$ - (1)

As the Equation (1) contains $z^{-1}p$ and $z^{-1}q$ we make the substitution Z = log z

$z^{-1}p=P$ and $z^{-1}q=Q$

Using there values in (1), it becomes

$P^2+Q^2=x^2+y^2$ (2)

As Eq. (2) dues not contain Z explicitly, we rewrite it as

$$P^2 - x^2 = y^2 - Q^2 = a^2, \text{ say } (3)$$

From (3)

$$P^2 = a^2 + x^2 \text{ and } Q^2 = y^2 - a^2$$

$$P = \sqrt{a^2 + x^2} \text{ and } Q = \sqrt{y^2 - a^2}$$

$$dz = Pdx + Qdy$$

$$dz = \sqrt{a^2 + x^2}dx + \sqrt{y^2 - a^2}dy$$

Integrating, we get

$$Z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sin h^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cos h^{-1}\left(\frac{y}{a}\right) + b$$

$\therefore$ the complete solution of (1) is

$$\log z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sin h^{-1}\left(\frac{x}{a}\right) + \frac{y}{a}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cos h^{-1}\left(\frac{y}{a}\right) + b$$

where a and b are arbitrary constants.

Singular solution does not exist and the General solution is found out as usual.

b. Solve the equation $\left(D^2 + D'^2\right)z = \sin 2x \sin 3y + 2\sin^2(x+y)$. **[CO 1 - H1]**

Solution:

The auxiliary equation is $m^2 + 1 = 0$

i.e., $m = \pm i$

$\therefore$ C.F. = $f(y+ix) + g(y-ix)$

$$PI_1 = \frac{1}{D^2 + D'^2}\sin 2x \sin 3y$$

$$= \frac{1}{D^2 + D'^2}\cdot\frac{1}{2}\{\cos(2x-3y) - \cos(2x+3y)\}$$

$$= \frac{1}{2}\left[\frac{1}{-4-9}\cos(2x-3y) - \frac{1}{-4-9}\cos(2x+3y)\right]$$

$$= \frac{-1}{13}\cdot\frac{1}{2}\{\cos(2x-3y) - \cos(2x+3y)\}$$

$$= \frac{-1}{13}\sin 2x \sin 3y$$

$$PI_2 = \frac{1}{D^2 + D'^2}2\sin^2(x+y)$$

$$= \frac{1}{D^2 + D'^2}\{1 - \cos(2x+2y)\}$$

$$= \frac{1}{D^2 + D'^2}(1) - \frac{1}{D^2 + D'^2}\cos(2x+2y)$$

$$= \frac{1}{D^2 + D'^2}e^{0x} - \frac{1}{-4-4}\cos(2x+2y)$$

$$= \frac{x^2}{2} + \frac{1}{8}\cos(2x+2y)$$

The complete solution is $Z = C.F. + PI_1 + PI_2$

i.e., $Z = f(y+ix) + g(y-ix) - \frac{1}{13}\sin 2x \sin 3y + \frac{x^2}{2} + \frac{1}{8}\cos(2x+2y)$

Problem 23

a. Solve $\sqrt{p} + \sqrt{q} = 1$. **[CO 1 - H2 - Apr/May 2015]**

Solution:

This is of the form $f(p,q) = 0$.

The complete integral is given by $z = ax + by + c$ where

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = \left(1-\sqrt{a}\right)^2$$

$\therefore$ The complete solution is $z = ax + \left(1-\sqrt{a}\right)^2 y + c$ - (1)

Differentiating partially w.r.t. c we get 0 = 1 (absurd)

$\therefore$ There is no singular integral

Taking $c = f(a)$ where f is arbitrary,

$$z = ax + \left(1-\sqrt{a}\right)^2 y + f(a) \quad - (2)$$

Differentiating partially w.r.t a, we get

$$0 = x + 2\left(1-\sqrt{a}\right)\left(\frac{-1}{2\sqrt{a}}\right)y + f'(a) \quad - (3)$$

Eliminating 'a' between (2) & (3) we get the general solution.

b. Solve $\left(x^2 - yz\right)p + \left(y^2 - zx\right)q = z^2 - xy$. **[CO 1 - H2 - Apr/May 2015]**

Solution:

This is a Lagrange's linear equation of the form $Pp + Qq = R$

The subsidiary equations are $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$ - (1)

$$\frac{dx - dy}{\left(x^2 - yz\right) - \left(y^2 - zx\right)} = \frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{dx - dz}{x^2 - yz - z^2 + xy}$$

i.e., $\frac{d(x-y)}{\left(x^2 - y^2\right) + z(x-y)} = \frac{d(y-z)}{y^2 - z^2 + x(y-z)} = \frac{d(x-z)}{x^2 - z^2 + y(1-z)}$

i.e., $\dfrac{d(x-y)}{(x-y)(x+y+z)}=\dfrac{d(y-z)}{(y-z)(x+y+z)}=\dfrac{d(x-z)}{(x-z)(x+y+z)}$ - (2)

i.e., $\dfrac{d(x-y)}{(x-y)}=\dfrac{d(y-z)}{y-z}=\dfrac{d(x-z)}{x-z}$ - (3)

Taking the first two ratios, and integrating $\log(x-y)=\log(y-z)+\log a$

$\therefore \dfrac{x-y}{y-z}=a$ - (4)

Similarly taking the last two ratios of (3) we get,

$\therefore \dfrac{y-z}{x-z}=b$ - (5)

But $\dfrac{x-y}{y-z}$ and $\dfrac{y-z}{x-z}$ are not independent solutions for $\dfrac{x-y}{y-z}+1$ gives $\dfrac{x-z}{y-z}$ which is the reciprocal of the second solution.

Therefore solution given by (4) and (5) are not independent. Hence we have to search for another independent solution.

Using multipliers x, y, z in equation (1) each ratio is $=\dfrac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$

Using multipliers 1, 1, 1 each ratio is $=\dfrac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$

$$\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}=\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{\frac{1}{2}d\left(x^2+y^2+z^2\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}=\frac{d(x+y+z)}{x^2+y^2+z^2-xy-yz-zx}$$

Hence $\frac{1}{2}d\left(x^2+y^2+z^2\right)=(x+y+z)d(x+y+z)$

Integrating $\frac{1}{2}\left(x^2+y^2+z^2\right)=\dfrac{(x+y+z)^2}{2}+k$

$\therefore \quad \left(x^2+y^2+z^2\right)=(x+y+z)^2+2k$

$x^2+y^2+z^2=x^2+y^2+z^2+2xy+2yz+2zx+2k$

i.e., $xy+yz+zx=b$

$\therefore$ the general solution is $\phi\left(xy+yz+zx,\dfrac{x-y}{y-z}\right)=0$

UNIT – II FOURIER SERIES

PART – A

1. Explain periodic function with example. [CO2 - L2]

Solution: A function $f(x)$ is said to have a period T if for all 'x' , $f(x+T)=f(x),$ where T is a positive period of $f(x)$.

Eg: $f(x)=\sin x$

$f(x+2\pi)=\sin(x+2\pi)=\sin x=f(x)$

$\therefore$ sin x is a periodic $f\underline{n}.$ with period 2π.

2. State Dirichiet's conditions in Fourier series. [CO2 – L1 - Nov/Dec 2016]

Solution:

(i) $f(x)$ is periodic, single-valued and finite.

(ii) $f(x)$ has a finite $n\underline{o}.$ of discontinuous in any one period.

(iii) $f(x)$ has atmost a finite $n\underline{o}.$ of maxima & minima.

3. Find 'bn' in the expansion of x^2 as a Fourier series in $(-\pi,\pi)$.
[CO2 – H1 - Nov/Dec 2014]

Solution:

Given: $f(x)=x^2,(-\pi,\pi)$

$f(-x)=x^2=f(x)$ $\Longrightarrow \boxed{f(-x)=f(x)}$

$\therefore$ The given $f\underline{n}'$ is Even

$\boxed{b_n = 0}$

4. If $f(x)$ is an odd $f\underline{n}.$ defined in $(-l,l)$, what are the values of a_o and a_n? [CO2 - H2]

Solution: $a_o = 0$ & $a_n = 0$ [$\because$ $f(x)$ is a_n odd $f\underline{n}.$]

5. Find the Fourier constants b_n for x sin x in $(-\pi,\pi)$. [CO2 – H1 - Apr/May 2013]

Solution: $\boxed{b_n = 0}$ ($\because$ x sin x is an Even fn)

6. Find constant term in the Fourier series expansion of $f(x)=x$ in $(-\pi,\pi)$.
[CO2 – H1]

Solution: $\boxed{a_o = 0}$ [$\because$ $f(x)=x$ is an odd fn in $(-\pi,\pi)$]

7. State Parseval's Identity for the half range cosine expansion of $f(x)$ in (0, 1).
[CO2 – L1]

Solution: Given: $\boxed{l=1}$ $2\int_0^1[f(x)]^2\,dx=\frac{a_o{}^2}{2}+\sum_{n=1}^{\infty}a_n{}^2$

Where: $a_o=2\int_0^1 f(x)\,dx$

$a_n=2\int_0^1 f(x)\cos n\pi x\,dx$

8. What do you mean by Harmonic Analysis? [CO2 - H2]

Solution: The process of finding the Fourier series for a $f\underline{n}.$ given by numerical value is known as Harmonic Analysis.

9. In the Fourier Expansion of $f(x) = \begin{cases} 1 + 2x/\pi, & (-\pi, 0) \\ 1 - 2x/\pi, & (0, \pi) \end{cases}$ Find the value of bn. [CO2 – H1]

Solution: $f(x) = 1 + \frac{2x}{\pi}$

$f(-x) = 1 - \frac{2x}{\pi} = f(x) \implies \boxed{f(-x) = f(x)}$

∴ The Given fn is Even

$\boxed{\text{bn} = 0}$

10. Obtain Fourier sine series for unity in (0, π). [CO2 - H2- Nov/Dec 2015]

Solution: HRSS $f(x) = \sum_{n=1}^{\infty} bn \sin nx$

$$\text{bn} = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 1. \sin nx \, dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi}$$

$$= -\frac{2}{\pi n}[\cos n\pi - \cos o] = -\frac{2}{\pi n}[(-1)^n - 1]$$

when 'n' is Even $\implies$ bn = 0

'n' is odd $\implies$ bn $= \frac{4}{\pi n}$

$$\boxed{\therefore f(x) = \sum_{n=1,3}^{\infty} \frac{4}{\pi n} \sin nx}$$

11. If the Fourier series for the $f\underline{n}.$ $f(x) = \begin{cases} 0, & 0 < x < \pi \\ \sin x, & \pi < x < 2\pi \end{cases}$

is $f(x) = \frac{-1}{\pi} + \frac{2}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \cdots\right] + \frac{1}{2}\sin x$ → (1)

Deduce that $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty = \frac{\pi - 2}{4}$ [CO2 – L3 - May/June 2015]

Solution: Put $x = {}^{\pi}/_{2}$ in (1)

$$f({}^{\pi}/_{2}) = \frac{-1}{\pi} + \frac{2}{\pi}\left[\frac{\cos {}^{2\pi}/_{2}}{1.3} + \frac{\cos {}^{4\pi}/_{2}}{3.5} + \cdots\right] + \frac{\sin\frac{\pi}{2}}{2}$$

$$0 = \frac{-1}{\pi} + \frac{2}{\pi}\left[\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty\right] + \frac{1}{2}$$

$$\frac{1}{\pi} = -\frac{2}{\pi}\left[\frac{1}{1.3} + \frac{1}{3.5} + \cdots\right] + \frac{1}{2}$$

$$\left(\frac{1}{\pi} - \frac{2}{\pi}\right)\left(\frac{-\pi}{2}\right) = \frac{1}{1.3} - \frac{1}{3.5} + \cdots$$

$$\frac{\pi-2}{4} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty$$

12. If the Fourier series of the $f\underline{n}.$ $f(x) = x + x^2, (-\pi < x < \pi)$ is $\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \left(\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right)$, find the value of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty$. [CO2 - H2]

Solution: Given:

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

Pu $\boxed{x = \pi}$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi - 0$$

$$\Rightarrow \quad \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}$$

$$\Rightarrow \quad \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty.$$

$f(x) = x + x^2, (-\pi, \pi)$

$x = \pi$

$f(-\pi) = -\pi + \pi^2$

$f(\pi) = \pi + \pi^2$

$\therefore \frac{f(-\pi)+f(\pi)}{2} = \pi^2$

13. If $f(x) = x^2 + x$ is expressed as a Fourier series in the interval $(-2, 2)$ to which value this series converges at $x = 2$. [CO2 - H2- Nov/Dec 2014]

Solution: $\because$ $x = 2$ (Point of continuity)

Given: $f(x) = x^2 + x$, (−2, 2)

$$f(-2) = 4 - 2 = 2$$

$$f(2) = 4 + 2 = 6$$

$$\therefore \frac{f(-2)+f(2)}{2} = \frac{2+6}{2} = 4.$$

14. Find the constant term in the Fourier series corresponding to $f(x) = cos^2 x$ expressed in the interval $(-\pi, \pi)$. [CO2 - H2]

Solution:

$$a_o = {}^1\!/_\pi \int_{-\pi}^{\pi} f(x)\,dx$$

$$= {}^1\!/_\pi \int_{-\pi}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = 1$$

$\boxed{cos^2 x = \frac{1+\cos 2x}{2}}$

$\therefore \boxed{a_o = 1}$

15. To which value, the half range sine series corresponding to $f(x) = x^2$ expressed in the interval (0, 2) converges at $x = 2$. [CO2 - H2]

Solution: Given: $f(x) = x^2$, (0, 2)

$x = 2$

$\therefore$ The Fourier series converges to

$$\boxed{f(x) = 4}$$

16. If $f(x) = \begin{cases} \cos x, & (0 < x < \pi) \\ 50, & (\pi < x < 2\pi) \end{cases}$ and $f(x) = f(x + 2\pi)$ for all x, find the sum of the Fourier series of $f(x)$ at $x = \pi$. [CO2 - H2]

Solution: Here $x = \pi$ (Pt. of discontinuity)

$$f(\pi) = \frac{\cos\pi + 50}{2} = \frac{(-1)+50}{2} = \frac{49}{2}$$

17. Find the root mean square of $f(x) = x^2$ in (0, π). [CO2 - H2]

Solution: Given: $f(x) = x^2$, (0, π).

RMS value $\bar{y}^2 = \frac{2}{\pi}\int_0^\pi [f(x)]^2\, dx$

$$= \frac{2}{\pi}\int_0^\pi (x^2)^2\, dx = \frac{2}{\pi}\int_0^\pi x^4\, dx$$

$$= \frac{2}{\pi}\left(\frac{x^5}{5}\right)_0^\pi = \frac{2}{\pi}\left(\frac{x^5}{5}\right) = \frac{2\pi^4}{5}$$

$$\therefore \boxed{\bar{y}^2 = \frac{2\pi^4}{5}}$$

18. Define Root Mean square of $f(x)$ over the range (a, b) [CO2 – L1]

Solution: The RMS value of $f(x)$ over the interval (a, b) is defined as

$$\text{RMS} = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

19. If $f(x) = 3x + 4x^3$, defined in the interval (–2, 2), then find the value of a_1 in the Fourier series Expansion. [CO2 - H2]

Solution: $f(x) = 3x - 4x^3$

$$f(-x) = -3x - 4x^3 = -(3x - 4x^3) = -f(x)$$

$$\therefore f(-x) = -f(x)$$

$\therefore f(x)$ is an odd $fn.$.

$$\therefore \boxed{a_n = 0} \qquad \boxed{a_1 = 0}$$

20. If the $fn.$ $x\cos x$ has the series expansion $\sum_{n=1}^{\infty} bn \sin nx$, find the value of b_1 in $(-\pi, \pi)$. [CO2 – L1]

Solution: $f(x) = \sum_{n=1}^{\infty} bn \sin nx$ [The given fn is odd. $\therefore a_0 = a_n = 0$]

$$bn = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^\pi x\cos x \sin nx\, dx$$

$$\therefore b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\frac{\sin 2x}{2}\, dx$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\, dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{-\cos 2x}{2}\right) - (1)\cancel{\left(\frac{-\sin 2x}{4}\right)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\cos 2\pi + 0\right] = \frac{\cancel{-\pi}}{\cancel{2}\pi}(1) = \frac{-1}{2}.$$

21. Find Half Range sine series for $f(x) = K$ in $0 < x < \pi$. [CO2 - H2]

Solution: HRSS $f(x) = \sum_{n=1}^{\infty} bn \sin nx$

$$bn = \frac{2}{\pi}\int_0^{\pi} K\sin nx\, dx = \frac{2K}{\pi}\int_0^{\pi}\sin nx\, dx$$

$$= \frac{2K}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{-2K}{n\pi}[\cos n\pi - \cos 0]$$

$$bn = \frac{-2K}{n\pi}[(-1)^n - 1]$$

When 'n' is even $\Rightarrow bn = 0$

'n' is odd $\Rightarrow bn = \frac{4K}{n\pi}$

$$\boxed{\therefore\ f(x) = \sum_{n=1,3}^{\infty}\left(\frac{4K}{\pi n}\right)\sin nx}$$

22. If $f(x) = 2x$ in (0, 4), then find the value of a_2 in the Fourier series Expansion. [CO2 – L1]

Solution: Here $2l = 4$ $\Rightarrow \boxed{l = 2}$

$$a_n = {}^1\!/_l\int_0^{2l} f(x)\frac{\cos n\pi x}{l}\, dx = {}^1\!/\!\cancel{2}\int_0^{4}\cancel{2}x\frac{\cos n\pi x}{2}\, dx$$

$$\therefore a_2 = \int_0^4 x\,\frac{\cos 2\pi x}{2}\, dx = \int_0^4 x\cos \pi x\, dx$$

$$= \left[(x)\left(\frac{\sin \pi x}{\pi}\right) - (1)\left(\frac{-\cos \pi x}{\pi^2}\right)\right]_0^4$$

$$= \frac{1}{\pi^2}[\cos 4\pi - \cos 0] = \frac{1}{\pi^2}(1-1) = 0$$

$$\therefore \boxed{a_2 = 0}$$

23. Write the Fourier series in complex form for $f(x)$ defined in the interval (C, C + 2π). [CO2 – L1]

Solution: $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$

Where $C_n = \frac{1}{2\pi}\int_C^{C+2\pi} f(x)\, e^{inx} dx$

24. State Parseval's Identity for expansion of $f(x)$ as fourier series in (0, 2l) [CO2 - H2- Nov/Dec 2014]

Solution:

$$\frac{1}{2l}\int_0^{2l}[f(x)]^2 = \frac{ao^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(an^2 + bn^2)$$

25. Does $f(x) = \tan x$ possess a fourier expansion? [CO2 – H1]

Solution:

$f(x) = \tan x$ has an infinite discontinuity. Dirichilet condition is not satisfied. Hence fourier Expansion does not exist.

26. Find the co-efficient C_n in the complex fourier series of $f(x) = e^{-x}, -1 < x < 1$. [CO2 - H2]

Solution:

$$C_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1} = -\frac{1}{2}(1+in\pi)\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-1}e^{-in\pi} - e^{1}e^{in\pi}\right]$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-1}(-1)^n - e^{1}(-1)^n\right]$$

$$= \frac{-1}{2(1+in\pi)}(-1)^n\left[e^{-1} - e^{1}\right]$$

$$= \frac{-(-1)^n}{\cancel{2}(1+in\pi)} . (\cancel{-}2\sin h1)$$

$$\boxed{C_n = \frac{(-1)^n \sin h}{(1+in\pi)}}$$

27. State Parseval's Identity for the half range cosine expansion of $f(x)$ in (0, 1). [CO2 – L1]

Solution: $f(x) = {}^{ao}/_{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$ $\quad (\because l = 1)$

Then $2\int_0^1 [f(x)]^2\, dx = \frac{ao^2}{2} + \sum_{n=1}^{\infty} an^2$

28. Find a_n in expanding e^{-x} as a fourier series in $(-\pi, \pi)$. [CO2 - H2]

Solution: Given: $f(x) = e^{-x}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-\pi}(-\cos n\pi) - e^{\pi}(-\cos n\pi)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[\cos n\pi\,(e^{\pi} - e^{-\pi})\right]$$

$$\boxed{a_n = \frac{(-1)^n}{\pi(1+n^2)}\,.\,2\sin h\pi}$$

29. If $f(x)$ is discontinuous at $x = a$, what does its Fourier series represent at that point.

[CO2 - H2]

Solution: $\frac{f(x-a)+f(x+a)}{2}$

30. Find the root mean square value of the $f\underline{n.}$ $f(x) = x$ in the interval $(0, l)$. [CO2 – L1- Nov/Dec 2015]

Solution: RMS value of $f(x)$ in $(0, l)$

$$\bar{Y}^2 = \frac{2}{l}\int_0^l [f(x)]^2\,dx = \frac{2}{l}\int_0^l \underline{x^2}\,dx$$

$$= \frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{2}{3l}(l^3) = \frac{2l^2}{3}$$

$$\therefore \bar{Y}^2 = \frac{2l^2}{3}.$$

PART B

1. Find the Fourier series of *f* (*x*) = x^2, (-π, π) & deduce that

i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6}$ ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\ldots = \frac{\pi^2}{12}$ iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \frac{\pi^2}{8}$

[CO2 - H2- Nov/Dec 2016]

Given:

$f(x) = x^2$ in $(-\pi, \pi)$

$f(-x) = (x)^2 = x^2 = f(x)$

$f(-x) = f(x)$

∴ The given function is even

$\therefore \boxed{bn = 0}$ $\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \text{an} \cos nx}$ → 1

To find:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}.\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$\Rightarrow \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$= \frac{2}{\pi}\left[2\pi.\frac{\cos nx}{n^2}\right] = \frac{2}{\pi}\left[2\pi.\frac{\cos nx}{n^2}\right]$ [∵ sin nπ = 0, sin 0 = 0]

$= \frac{2}{\pi}\left[2\pi.\frac{(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$ [∵ cos nπ = $(-1)^n$]

=> $\boxed{a_n = \frac{4(-1)^n}{n^2}}$

(1) => $f(x) = \frac{2\pi^2}{3 \times 2} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$

$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$

$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$ → 2

i) Put $x = \pi$ in (2)

(2) => $\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$

$f(x) = x^2$
$f(\pi) = \pi^2$
$f(-\pi) = (-\pi)^2 = \pi^2$
$f(x) = \frac{f(\pi) + f(-\pi)}{2} = \frac{2\pi^2}{2} = \pi^2$
$f(x) = \pi^2$

$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}$

$\frac{3\pi^2 - \pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$

$\frac{2\pi^2}{3}.\frac{1}{4_2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ [∵ $(-1)^{2n} = 1$]

$f(x) = x^2$
$f(0) = 0$

$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

$\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.\right] = \frac{\pi^2}{6}$ → 3

ii) Put $x = 0$ in (2)

(2) => $0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos 0$

$\frac{-\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ [∵ cos 0 = 1]

$\frac{-\pi^2}{3} = 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots.\right]$

To find:

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x + x^2)dx$

$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$

$= \frac{1}{\pi}.\frac{2\pi^3}{3} = \frac{2\pi^2}{2}$

$\boxed{a_0 = \frac{2\pi^2}{3}}$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$

$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{\sin nx}{n}\right) - (1 + 2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[+(1+2\pi)\left(\frac{+\cos n\pi}{n^2}\right) - (1-2\pi)\left(\frac{-\cos n(-\pi)}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[(1+2\pi)\frac{(-1)^n}{n^2} - (1-2\pi)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{(-1)^n}{\pi n^2}[1 + 2\pi - 1 + 2\pi] = \frac{(-1)^n}{\pi n^2}.\, 4\pi = \frac{4(-1)^n}{n^2}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\,(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots.\right]$$

$$\frac{\pi^2}{3} = \frac{1}{4}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots.\right]$$

$$\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots.\right] = \frac{\pi^2}{12} \qquad \rightarrow 4$$

iii) $\mathbf{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots. = \frac{\pi^2}{8}}$

Add eqn. (3) and (4)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.\right] + \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.\right]$$

$$= 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right]$$

$$\frac{2\pi^2 + \pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right]$$

$$\frac{3\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right]$$

$$\frac{\pi^2}{4}.\frac{1}{2} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right]$$

$$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right] = \frac{\pi^2}{8}$$

2. Find the Fourier series of $f\,(x) = x + x^2$, $(-\pi, \pi)$ & deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots. = \frac{\pi^2}{6}$. [CO2 - H2- Nov/Dec 2014]

Given:

$f\,(x) = x + x^2$, $(-\pi, \pi)$

$f\,(-x) = -x + x^2 = x + x^2$

$\therefore$ The given function is neither even nor odd

$$\boxed{f\,(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

Given:

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

Pu $\boxed{x = \pi}$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi - 0$$

$$\Rightarrow \quad \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}$$

$$\Rightarrow \quad \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty.$$

$f(x) = x + x^2, (-\pi, \pi)$

$x = \pi$

$f(-\pi) = -\pi + \pi^2$

$f(\pi) = \pi + \pi^2$

$\therefore \frac{f(-\pi)+f(\pi)}{2} = \pi^2$

3. Find the Fourier series of *f* (*x*) = $|x|$, (-π, π) & deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Given: **[CO2 - H2- May/June 2014]**

$f(x) = |x|, (-\pi, \pi)$

$f(-x) = |x| = f(x)$

$\therefore$ The given function is even

$\therefore$ $\boxed{b_n = 0}$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx} \quad \rightarrow 1$$

To find:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}.\frac{\pi^2}{2} = \pi$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\boxed{a_n = \frac{2}{\pi n^2}[(-1)^n - 1]}$$

When 'n' is even => an = $\frac{2}{\pi n^2}$ [1 – 1] = 0

When 'n' is odd => an = $\frac{2}{\pi n^2}$ [-1 – 1] = $\frac{-4}{\pi n^2}$

(1) => $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3}^{\infty}\frac{1}{n^2}\cos nx$ → 2

$= \frac{1}{\pi}\left[-(\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right) + 2\frac{\cos n\pi}{n^3} + (-\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right) - 2\frac{\cos n\pi}{n^3}\right]$

$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n^3} + (-\pi+\pi^2)\frac{\cos n\pi}{n^3}\right]$

$= \frac{\cos n\pi}{n\pi}$ [-π – π² – π + π²] = $\frac{\cos n\pi}{n\pi}$ (-2π) = $\frac{-2(-1)^n}{n}$

$$\boxed{b_n = \frac{-2(-1)^n}{n}}$$

(1) => $f(x) = \frac{2\pi^2}{2 \times 3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$ → 2

Put x = π in (2)

(2) => $\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 0$ [∵ sin nπ = 0]

$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$ [cos nπ = $(-1)^n$]

$\frac{3\pi^2 - \pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$

$\frac{2\pi^2}{3}.\frac{1}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ [$(-1)^n = 1$]

$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

$\left[\frac{1}{1^2} + \frac{1}{2} + \frac{1}{3^2} + \ldots.\right] = \frac{\pi^2}{6}$

$f(x) = x + x^2$
$f(-\pi) = -\pi + \pi^2$
$f(\pi) = \pi + \pi^2$
$f(x) = \frac{f(-\pi) + f(\pi)}{2}$
$= \frac{-\pi + \pi^2 + \pi + \pi^2}{2}$
$f(x) = \frac{2\pi^2}{2} = \pi^2$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\pi - \frac{2x}{\pi}\right)\cos nx\,dx$

$= \frac{2}{\pi}\left[\left(\pi - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\left(\frac{-2}{\pi}\right)\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\frac{-2}{\pi}\left[\frac{\cos n\pi}{n^2}\right] - \left(\frac{-2}{\pi}\right)\left[\frac{\cos 0}{n^2}\right]\right]$ [cos nπ = $(-1)^n$]

$= \frac{2}{\pi}\left[\frac{-2(-1)^n}{\pi n^3} + \frac{2}{\pi n^3}\right] = \frac{4}{\pi^2 n^3}[1 - (-1)^n]$

$$\boxed{a_n = \frac{4}{\pi^2 n^3}[1 - (-1)^n]}$$

When 'n' is even $a_n = \frac{4}{\pi^2 n^3}[1 - 1] = 0$

When 'n' is odd $a_n = \frac{4}{\pi^2 n^3}[1 + 1] = \frac{8}{\pi^2 n^3}$

(1) => $f(x) = \frac{2(\pi-1)}{2} + \frac{8}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}\cos nx$

$f(x) = (\pi - 1) + \frac{8}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}\cos nx$ → 2

Put $x = 0$ in (2)

(2) => $\pi = (\pi - 1) + \frac{8}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}(1)$ [cos 0 = 1]

$\pi - \pi + 1 = \frac{8}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}$

$1.\frac{8}{\pi^2} = \sum_{n=1,3}^{\infty}\frac{1}{n^2}$

$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right] = \frac{\pi^2}{8}$

Put $x = 0$ in (2)

$\boxed{\begin{array}{l} f(x) = |x| \\ f(0) = 0 \end{array}}$

(2) => $0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3}^{\infty}\frac{1}{n^2}\cos 0$

$\frac{\pi}{2} = \frac{-4}{\pi}\sum_{n=1,3}^{\infty}\frac{1}{n^2}$ [∵ cos 0 = 1]

$\frac{\pi^2}{8} = \sum_{n=1,3}^{\infty}\frac{1}{n^2}$

$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right] = \frac{\pi^2}{8}$

4. Find the Fourier series of $f(x) = \begin{cases} \pi - \frac{2x}{\pi}, (-\pi, 0) \\ \pi + \frac{2x}{\pi}, (-\pi, 0) \end{cases}$ & deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. [CO2 - H2]

Given:

$f(x) = \begin{cases} \pi - \frac{2x}{\pi}, (-\pi, 0) \\ \pi + \frac{2x}{\pi}, (-\pi, 0) \end{cases}$

$f(x) = \pi - \frac{2x}{\pi}$

$f(-x) = \pi - \frac{2(-x)}{\pi} = \pi + \frac{2x}{\pi} = f(x)$

∴ The given function is an even function

∴ $\boxed{b_n = 0}$ $\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos nx}$ → 1

To find:

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(\pi - \frac{2x}{\pi}\right)dx$

$= \frac{2}{\pi}\left[\pi x - \frac{2x^2}{2\pi}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{\pi}\right] = \frac{2}{\pi}[\pi^2 - \pi]$

$= \frac{2\pi}{\pi}[\pi - 1] = 2[\pi - 1]$

$\boxed{a_0 = 2[\pi - 1]}$

$a_0 = \frac{l}{n^2\,\pi^2}[1 - (-1)^n]$

When 'n' is even => $a_n = \frac{l}{\pi^2 n^3}[1 - 1] = 0$

When 'n' is odd => $a_n = \frac{l}{\pi^2 n^3}[1 + 1] = \frac{2l}{\pi^2 n^3}$

$b_0 = \frac{1}{l}\int_0^l f(x)\, sin\frac{n\pi x}{l}dx + \frac{1}{l}\int_l^{2l} f(x)\, sin\frac{n\pi x}{l}dx$

$= \frac{1}{l}\int_0^l f(l-x)\, sin\frac{n\pi x}{l}dx + 0$

$= \frac{1}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$

$= \frac{1}{l}\left[0 - \left(-l.\frac{\cos 0}{\frac{n\pi}{l}}\right)\right] = \frac{1}{l}\left[\frac{l^2}{n\pi}\right] = \frac{l}{n\pi}$

$\boxed{b_n = \frac{l}{n\pi}}$ [cos 0 = 1]

(1) => $f(x) = \frac{l}{2 \times 2} + \frac{2l}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$

$f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$ → 2

Put x = 0 in (2)

$\frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}(\cos 0) + 0$

$\frac{l}{2} - \frac{l}{4} = \frac{2l}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}$ [∵ sin 0 = 0]

$\frac{l}{4} = \frac{2l}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}$

$\frac{l}{4}.\frac{2l}{\pi^2} = \sum_{n=1,3}^{\infty}\frac{1}{n^2}$

$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] = \frac{\pi^2}{8}$

5. Find the Fourier series of *f* (*x*) = $\begin{cases}(l-x), (0, l)\\ 0 \qquad , (l, 2l)\end{cases}$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Given: **[CO2 - H2]**

$f(x) = \begin{cases}(l-x), (0, l)\\ 0 \qquad , (l, 2l)\end{cases}$

→ 1

$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} bn\sin\frac{n\pi x}{l}}$

To find:

$$a_0 = \frac{1}{l}\int_0^l f(x)dx$$

$$= \frac{1}{l}\int_0^l f(x)\,dx + \frac{1}{l}\int_l^{2l} f(x)\,dx + \frac{1}{l}\int_0^l (l-x)\,dx + 0$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l \qquad [\because f(x) = 0 \text{ in } (l, 2l)]$$

$$= \frac{1}{l}\left[l^2 - \frac{l^2}{2}\right] = \frac{1}{l}\left[l^2 - \frac{2l^2 - l^2}{2}\right] = \frac{1}{l}.\frac{l^2}{2} = \frac{l}{2}$$

=> $\boxed{a_0 = \frac{l}{2}}$

$$a_n = \frac{1}{l}\int_0^l f(x)\, con\frac{n\pi x}{l}dx + \frac{1}{l}\int_l^{2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\int_0^l f(l-x).\cos\frac{n\pi x}{l}dx + 0$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} + \frac{\cos 0}{\frac{n^2\pi^2}{l^2}}\right] = \frac{1}{l}\left[\frac{l^2}{n^2\pi^2}\left(-(-1)^n + 1\right)\right]$$

$$a_n = \frac{4}{\pi^2 n^3}[1 - (-1)^n]$$

6. Find the Fourier series of $f(x) = \begin{cases} 1+x, (-2,0) \\ 1-x, (0,2)\end{cases}$ hence deduce that $\sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} bn\sin\frac{n\pi x}{l}$ [CO2 - H2-Apr/May 2016]

Given:

$$f(x) = \begin{cases} 1+x, (-2,0) \\ 1-x, (0,2)\end{cases}$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} bn\sin\frac{n\pi x}{l}} \qquad \rightarrow 1$$

$f(x) = 1 + x$

$f(-x) = 1 - x = f(x)$

$\therefore$ The given function is even

$\boxed{\therefore b_n = 0}$

To find:

$$a_0 = \frac{2}{l}\int_0^l f\,(x)dx = \frac{2}{2}\int_0^2 f\,(1-x)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^2 = 2 - \frac{4^2}{2} = 0$$

$$a_0 = 0$$

$$a_n = \frac{2}{l}\int_0^l f\,(x)\,con\frac{n\pi x}{l}dx$$

$$= \frac{2}{2}\int_0^2 f\,(1-x)\,cos\frac{n\pi x}{l}dx$$

$$= \left[(1-x)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\,\pi^2}{4}}\right)\right]_0^2$$

$$= \frac{4}{n^2\,\pi^2}\left[-cos\ \frac{n\pi 2}{2} + \cos 0\right] = \frac{4}{n^2\,\pi^2}\,[-(-1)^n + 1]$$

$$\boxed{a_n = \frac{4}{\pi^2 n^3}\,[1-(-1)^n]}$$

$$a_n = \frac{4}{\pi^2 n^3}\,[1-(-1)^n]$$

When 'n' is even => $a_n = \frac{4}{\pi^2 n^3}$ [1 – 1] = 0

When 'n' is odd => $a_n = \frac{4}{\pi^2 n^3}$ [1 + 1] = $\frac{8}{\pi^2 n^3}$

(1) => $f\,(x) = \sum_{n=1,3}^{\infty}\frac{8}{\pi^2 n^3}\cos\frac{n\pi x}{2}$ → 2

Put x = 0 in (2)

(2) => $1 = \frac{8}{\pi^2}\sum_{n=1,3}^{\infty}\frac{\cos 0}{n^2}$

$1 = \frac{8}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}$ $[\cos 0 = 1]$

$\sum_{n=1,3}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$

$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right] = \frac{\pi^2}{8}$

$f\,(x) = 1 - x$
$f\,(0) = 1$
$f\,(0) = 1$
$f\,(x) = \frac{f\,(0) + f\,(0)}{2}$
$f\,(x) = 1$

7. Find the half range sine series for $f\,(x) = x\,(\pi - x)$ in (0, π) deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots. = \frac{\pi^3}{3^2}$. **[CO2 - H2-Nov/Dec 2015]**

Given:

$f\,(x) = x\,(\pi - x)$ in (0, π)

HRSS: $\boxed{f\,(x) = \sum_{n=1}^{\infty} bn \sin nx}$ → 1

To find:

$$b_0 = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi f(\pi x - x^2)\sin nx\,dx$$
$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

To find:

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\,dx$$
$$= \frac{2}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\left[\frac{3\pi^3 - 2\pi^3}{6}\right]$$
$$= \frac{2}{\pi}\left[\frac{\pi^3}{6}\right] = \frac{\pi^3}{3}$$

$$\boxed{a_0 = \frac{\pi^3}{3}}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\cos nx\,dx$$
$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$
$$= \frac{2}{\pi}\left[(\pi - 2\pi)\left(\frac{\cos n x}{n^2}\right) - (\pi)\left(\frac{\cos 0}{n^2}\right)\right]$$
$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n^2} - \pi\frac{1}{n^2}\right] = -\frac{2\pi}{\pi n^2}\left[\frac{\pi^3}{6}\right][(-1)^n + 1]$$

$$\boxed{a_n = \frac{-2}{n^3}[(-1)^n + 1]}$$

When 'n' is even => $a_n = \frac{-2}{n^3}[1 + 1] = \frac{4}{n^3}$

When 'n' is odd => $a_n = \frac{-2}{n^3}[1 - 1] = 0$

(1) => $f(x) = \frac{\pi^2}{3 \text{ x } 2} - 4\sum_{n=2,4}^{\infty}\frac{1}{n^3}\cos nx$

$$f(x) = \frac{\pi^2}{6} - 4\sum_{n=2,4}^{\infty}\frac{1}{n^3}\cos nx \qquad \rightarrow 2$$

$$b_n = \frac{2}{\pi}\left[(\pi^2 - \pi^2)\left(\frac{-\cos n\pi}{n^2}\right) - 0 - 2\left(\frac{\cos n\pi}{n^3}\right) + 2\left(\frac{\cos 0}{n^3}\right)\right]$$
$$= \frac{2}{\pi}\left[\frac{-2}{n^3}(-1)^n + \frac{2}{n^3}\right] = \frac{2}{\pi}\left[\frac{2}{n^3}(1 - (-1)^n)\right]$$

$$\boxed{b_n = \frac{4}{\pi n^3}[1 - (-1)^n]}$$

When 'n' is even => $b_n = \frac{4}{\pi n^3}[1 - 1] = 0$

When 'n' is odd => $b_n = \frac{4}{\pi n^3}[1 + 1] = \frac{8}{\pi n^3}$

(1) => $f(x) = \frac{8}{\pi}\sum_{n=1,3}^{\infty}\frac{1}{n^3}\sin nx \qquad \rightarrow 2$

Put $x = \frac{\pi}{2}$ in (2)

(2) => $\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{n=1,3}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left[\frac{\sin\frac{\pi}{2}}{1^3} + \frac{\sin\frac{3\pi}{2}}{3^3} + \ldots\right]$$
$$\frac{\pi^2}{4}.\ \frac{\pi}{8} = \left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots.\right]$$
$$\frac{\pi^3}{32} = \left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

$\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right] = \frac{\pi^3}{32}$ $\left[\because sin\frac{\pi}{2} = 1\ sin\frac{3\pi}{2} = -1\ sin\frac{5\pi}{2} = 1\ sin\frac{7\pi}{2} = -1\right]$

8. Compute the first two harmonics of the Fourier series for *f* (*x*) from the following data. [CO2 - H2-Apr/May 2015]

x	0	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x)$ → 1

x	**y**	**Cos *x***	**Cos 2*x***	**Y cos *x***	**Y cos 2*x***	**Sin *x***	**Sin 2*x***	**Y sin *x***	**Y sin 2*x***
0	1.0	1	1	1.0	1	0	0	0	0
$\frac{\pi}{3}$	1.4	0.5	-0.5	0.7	-0.7	0.866	0.866	1.212	1.212
$\frac{2\pi}{3}$	1.9	-0.5	-0.5	-0.95	-0.95	0.866	-0.866	1.6454	-1.6454
π	1.7	-1		-1.7	1.7	0	0	0	0
$\frac{4\pi}{3}$	1.5	-0.5	-0.5	-0.75	-0.75	-0.866	0.866	-1.299	1.299
$\frac{5\pi}{3}$	1.2	0.5	-0.5	0.6	-0.6	-0.866	-0.866	-1.0392	-1.03
	8.7			-1.1	-0.3			0.5192	-0.1644

PARSEVEL'S IDENTITY

HRCS is $(0, \pi)$, $\frac{2}{\pi}\int_0^{\pi}[F(x)]^2\ dx = \frac{a_0^2}{2}\sum_{n=1}^{\infty} a_0^2$ → 3

Sub. $f(x) = x(\pi - x)$, $a_0 = \frac{\pi^2}{3}$, $a_n = \frac{-4}{n^2}$ in equ (3)

$$(3) \Rightarrow \frac{2}{\pi}\int_0^{\pi}(\pi x - x)^2 dx = \left(\frac{\pi^2}{3}\right)^2\frac{1}{2} + \sum_{n=2,4}^{\infty}\left(\frac{-4}{n^2}\right)^2$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi^2x^2 + x^4 - 2\pi x^3)dx = \frac{\pi^4}{18} + 16\sum_{n=2,4}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}\left[\frac{\pi^2x^2}{3} + \frac{x^5}{5} - \frac{2\pi x^4}{4}\right]_0^{\pi} = \frac{\pi^4}{18} + 16\sum_{n=2,4}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{3} + \frac{\pi^5}{5} - \frac{\pi^5}{4}\right] = \frac{\pi^4}{18} + 16\sum_{n=2,4}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}\left[\frac{10\pi^5 + 6\pi^5 - 15\pi^5}{30}\right] = \frac{\pi^4}{18} + 16\sum_{n=2,4}^{\infty}\frac{1}{n^4}$$

$\frac{2}{\pi}\left[\frac{\pi^5}{30}\right] = \frac{\pi^4}{18} + 16\left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ ...\right]$

$\frac{\pi^4}{15} - \frac{\pi^4}{18} = 16\left[\frac{1}{16} + \frac{1}{16.16} + \frac{1}{16.81} + \ ...\right]$

$\frac{3\pi^4}{270} = \frac{16}{16} = \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ ...\right]$

$\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ ...\right] = \frac{\pi^4}{90}$

$a_0 = 2\left[\frac{\sum Y}{n}\right] = 2\left[\frac{8.7}{6}\right] = 2 \text{ x } 1.45 = 2.9$

$\boxed{a_0 = 2.9}$

$a_1 = 2\left[\frac{\sum Y \cos x}{n}\right] = 2\left[\frac{-1.1}{6}\right] = -0.366$

$\boxed{a_1 = 0.366}$

$a_2 = 2\left[\frac{\sum Y \cos 2x}{n}\right] = 2\left[\frac{-0.3}{6}\right] = -0.1$

$\boxed{a_2 = -0.1}$

$b_1 = 2\left[\frac{\sum Y \sin x}{n}\right] = 2\left[\frac{0.5192}{6}\right] = 0.1730$

$\boxed{b_1 = -0.1}$

$b_2 = 2\left[\frac{\sum Y \sin 2x}{n}\right] = 2\left[\frac{-0.1644}{2}\right] = -0.0548$

$\boxed{b_2 = -0.0548}$

(1) => $f(x) = \frac{1.45}{2} + (-0.36 \cos x + 0.17 \sin x) + (-0.1 \cos 2x - 0.054 \sin 2x)$

$f(x) = 1.45 + (-0.36 \cos x + 0.17 \sin x) + (-0.1 \cos 2x - 0.054 \sin 2x)$

9. Find the Fourier series of second harmonic to represent the function given in the following data. [CO2 - H2]

x	0	1	2	3	4	5
y (*x*)	9	19	24	28	26	20

Here the length of the interval is 6.

$2l = 6,\ l = 3$

$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l}$

$$\boxed{f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l}}$$ → 6

x	$\frac{\pi x}{3}$	$\frac{2\pi x}{3}$	**y**	**cos** $\frac{\pi x}{3}$	**y cos** $\frac{\pi x}{3}$	**y cos** $\frac{2\pi x}{3}$	**sin** $\frac{\pi x}{3}$	**y sin** $\frac{\pi x}{3}$	**y sin** $\frac{2\pi x}{3}$
0	0	0	9	1	9	9	0	0	0
1	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	18	0.5	9	-9	0.866	15.58	15.6
2	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	24	-0.5	-12	-24	0.866	20.78	0
3	π	2π	28	-1	-28	28	0	0	0
4	$\frac{4\pi}{3}$	$\frac{8\pi}{3}$	26	-0.5	-13	-13	-0.866	-22.51	22.6
5	$\frac{5\pi}{3}$	$\frac{10\pi}{3}$	20	0.5	10	-10	-0.866	-17.32	-17.4
	125			-25	-19			-3.4	20.8

10. Find the Fourier series of $f(x) = x(\pi - x)^2$ in $(0, 2\pi)$ & deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$ [CO2 - H2-Nov/Dec 2012]

Given:

$f(x) = x(\pi - x)^2$ in $(0, 2\pi)$

$$\boxed{f(x) = \frac{a_0}{2} \sum_{n=1}^{\infty} an \cos nx + \sum_{n=1}^{\infty} bn \sin nx}$$ → 1

To find:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} f(\pi - x)^2 dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{2\pi^3}{3}$$

$$a_0 = \frac{2\pi^3}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx = \frac{1}{\pi}\int_0^{2\pi} f(\pi - x)^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\pi\frac{\cos 2n\pi}{n^2} + 2\pi\frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}[2\pi(1+1)] = \frac{4\pi}{\pi n^2}$$ [cos 2nπ = 1, cos 0 = 1]

$$\boxed{a_0 = \frac{4}{n^2}}$$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_0^{2\pi} f(\pi - x)^2 \sin nx\, dx$

$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$

$a_0 = 2\left[\frac{\sum Y}{n}\right] = 2\left[\frac{125}{6}\right] = 41.66$

$\boxed{a_0 = 41.66}$

$a_1 = 2\left[\frac{\sum Y \cos\frac{\pi x}{3}}{n}\right] = 2\left[\frac{-25}{6}\right] = -8.33$

$\boxed{a_1 = -8.33}$

$a_2 = 2\left[\frac{\sum Y \cos\frac{2\pi x}{3}}{n}\right] = 2\left[\frac{-19}{6}\right] = -6.33$

$\boxed{a_2 = -6.33}$

$b_1 = 2\left[\frac{\sum Y \sin\frac{\pi x}{3}}{n}\right] = 2\left[\frac{-3.4}{6}\right] = -1.13$

$\boxed{b_1 = -1.13}$

$b_2 = 2\left[\frac{\sum Y \sin\frac{2\pi x}{3}}{n}\right] = 2\left[\frac{20.8}{6}\right] = 6.9$

$\boxed{b_2 = 6.9}$

(1) => $f(x) = \frac{41.66}{2} + (-8.33 \cos\frac{\pi x}{3} - 1.13 \sin\frac{\pi x}{3}) + (-6.33\cos\frac{2\pi x}{3} + 6.9 \sin\frac{2\pi x}{3})$

$f(x) = 20.83 + (-8.33\cos\frac{\pi x}{3} - 1.13\sin\frac{\pi x}{3}) + (-6.33\cos\frac{2\pi x}{3} + 6.9\sin\frac{2\pi x}{3})$.

UNIT – III APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

PART –A

1. **Classify the p.d.e** $\mathbf{x^2u_{xx} + 2xyu_{xy} + (1+y^2)u_{yy} - 2u_x = 0}$. **[CO3 - L2-Apr/May 2015]**

Ans: $A = x^2, B = 2xy \quad , C = 1 + Y^2$

$\Delta = B^2 - 4AC = -4x^2 < 0$

$\therefore$ p.d.e is elliptic.

2. Classify $\mathbf{u_{xx} = U_{yy}}$ **. [CO3 - L2 - May/June 2016]**

Ans : $A = 1, B = O, C = -1$

$\Delta = B^2 - 4AC => 0$

$\therefore$ p.d.e is hyperbolic.

3. Classify $\mathbf{4u_{xx} + 4U_{xy} + U_{yy} - 6u_x - 8u_y - 16u = 0.}$ **[CO3 - L2]**

Ans :$A = 4, B = 4, C - 1$

$\Delta = 0$

$\therefore$ p.d.e is parabolic

4. Classify the p.d.e $\mathbf{u_{xx} + 4U_{xy} + 4u_{yy} - 12u_x + u_y + 7u = x^2 + y^2}$ **. [CO3 - L2]**

Ans: $A = 1, B = 4, C = 4$

$\Delta = B^2 - 4AC = 0$

$\therefore$ p.d.e is parabolic.

5. In wave equations $\mathbf{\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial t^2}}$ **what is the physical meaning for** $\mathbf{C^2}$**and What is the agent for vibration**? **[CO3 - H2]**

$$\text{Ans : } C^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length}}$$

Tension is the agent for vibration.

6. Write the one dimensional heat equation. [CO3 – L1]

$$\text{Ans}:\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$$

7. What is the Fourier law of heat conduction. [CO3 - H2]

$$\text{Ans}:Q=-kA\left(\frac{\partial u}{\partial x}\right)_x$$

$$Q=\text{Quantity of heat fliwing}$$

$$k=\text{thermal conductivity}$$

$$A=\text{area of cross section}$$

$$\frac{\partial u}{\partial x}=\text{temperature gradient}$$

(The rate at which heat flows across an area A at distance x from one end of a bar is proportional to temperature gradient.)

8. Write all possible solutions for O.D.H.E. [CO3 – L1- May/June 2016]

$$\text{Ans: (i)}u(x,t)=(C_1e^{kx}+C_2e^{-kx})C_{3e}\propto^{2K^2t}$$

$$\text{(ii) }u(x,t)=(C_1\cos kx+C_2\sin kx)e^{-}\propto^2k^2t$$

$$\text{(iii)}u(x,t)=C_1x+C^2$$

9. A rod of length l is kept at T_1^0 and T_2^0at the ends x=0 and x=1, the initial temperature distribution is u_0, formulae the mathematical model. [CO3 – L3]

$$\text{Ans}:\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$$

$$u(o,t)=T_1,u\ (0,t)=T_2,u\ (x,o)=u_0$$

10. A rod of length 20cm whose one end is kept at 30^{0}C and the other end at 70^{0}C, until steady state prevails, find the steady state prevails, find the steady state temperature. [CO3 - H2]

Ans : $u = ax + b$

$$x = o, u = 30, b = 30$$

$$x = 20, u = 70, a = \frac{40}{20} = 2$$

$$u = 2x + 30, \text{this is the steady state temp distribution}$$

11. A bar of length 50 cm has its ends kept at 20^0C and 100^0C until stead state prevails. Find the temperature t any point. [CO3 - H2]

$$\text{Ans} : u(x) = 1.6x + 20$$

12. A rod 30 cm long its ends A and B kept at 20^0C and 80^0C until stead state prevails. Find the temperature t any point. [CO3 - H2]

$$\text{Ans} : u = 2x + 20$$

13. State the two – dimensional Laplace equation. [CO3 – L1]

$$\text{Ans} : \frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2} = 0$$

14. Write three possible solutions o Laplace equations in two dimensions. [CO3 – L1-Nov/Dec 2015]

$$\text{Ans} : (i) u(x,y) = \left(C_1 e^{kx} + C_2 e^{-kx}\right)(C_3 \cos ky + C_4 \sin ky)$$

$$(ii) u(x,y) = \left(C_1 \cos kx + C_2 \sin kx\right)(C_3 e^{ky} + C_4 e^{-ky})$$

$$u(x,y) = (c_1 x + C_2)(C_3 y + C_4)$$

15. In 2D heat equation or Laplace equation, What is the basic assumption. [CO3 - H2]

Ans : When the heat flow is along curves instead of stratight lines, the curves lying in parallel planes the flow is called two dimensional.

16. A rectangular plate is bounded by the lines x=o, y=o, x=a, y=b. Its surfaces are insulated. The temperature along x=0, y=0 are kept at 0^0C, other sides are at 100^0 C. Write the B.C's . [CO3 – L3]

$$\text{Ans: } \nabla^2 u = 0$$

$$\text{(i)} u(0, y) - 0$$

$$\text{(ii)} u(x, 0) = 0$$

$$\text{(iii)} u(a, y) = 100$$

$$\text{(iv)} u(x, a) = 100$$

17. In steady state conditions derive the solution of one dimenslonal heat flow equation. [CO3 – L3]

Ans. One dimensional heat equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\text{under steady state Ans : } \frac{\partial u}{\partial t} = 0$$

$$\therefore (1) \text{becomes, Ans : } \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Longrightarrow u = ax + b.$$

18. In one dimensional heat equation $\mathbf{u_t = a^2 u_{xx}}$. What does $\mathbf{a^2}$ stands for ?
[CO3 – L1-May/June 2013]

Ans. $A^2 = \text{Thermal diffusivity.}$

19. A tightly stretched string of length 2L is fixed at both ends. The mid point of the string is displaced to a distance ‘b’ and released from rest in this position write the Initial conditions. [CO3 – L1]

Ans. The initial conditions are

$$\text{(i)} t = o, \left(\frac{\partial y}{\partial t}\right) = 0$$

$$\text{(ii) } t = 0, f(x) = \{ \frac{bx}{L'} \quad 0 < x < L$$
$$\frac{b}{L} \ (2L - x), L < x < 2L)$$

20. Write the initial conditions of the wave equation if the string has an initial displacement. [CO3 – L1-May/June 2014]

$$\text{Ans.} \left(\frac{\partial y}{\partial t}\right)_{t=o} = 0$$

$$y(x, 0) = f(x)$$

21. State one dimensional heat equation with the initial and boundary conditions.

[CO3 – L1]

Ans. The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

$$u(l, t) = k_1^0 C\ V\ t > o$$

$$u(l, t) = k_2^0 C\ V\ t > O$$

$$u(x, 0) = f(x) \text{in } (0, l)$$

22. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string is subjected to initial displacement f (x) and initial velocity g (x) [CO3 – L1]

Ans. $(A) y\ (0, t) = 0$

$$(b) y(l, t) = 0$$

$$(c) \frac{\partial y(x, 0)}{\partial t} = g(x)$$

$$(d)\ y\ (x, 0) = f(x)$$

23. Solve the equation $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, given that $u(x, 0) = 4_e^{-x}$ by the method of separation of variables. [CO3 - H2]

Ans. We know that $u(x, y) = X(x)Y(y) = XY$

$$\frac{\partial u}{\partial x} = X' \, Y \frac{\partial u}{\partial y} = XY'$$

$$\therefore 3X'Y + 2XY' = 0$$

$$\Rightarrow \quad \frac{3X'}{X} + \frac{2Y'}{Y} = 0$$

$$\Rightarrow \frac{3X'}{X} = -\frac{2Y'}{Y} = k(\text{say})$$

24. Write the initial conditions of the wave equation if the string has an initial displacement but no initial velocity. [CO3 – L1]

$$\text{Ans.} \left[\frac{\partial y}{\partial t}\right]_{t=0} = 0$$

$$y(x, 0) = f(x)$$

25. Classify the partial differential equation [CO3 - L2]

$$\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial^2 u}{\partial x \, \partial y} + \frac{\partial^2 u}{\partial y^2} = e^{2x+3y}$$

$$\text{Ans.} A = 1, B = 2, C = 1$$

$$\Delta = B^2 - 4AC$$

$$= 4 - 4 = 0$$

$$\therefore \text{The p.d.e is parabolic.}$$

26. A infinitely long uniform plate is bounded by the edges x=0, x=l and an end right angles to them. The breadth of the edge y = 0 is l and is maintained at f(x). All the other edges are kept at 0^0C. Write down the boundary condition in mathematical form. [CO3 – L1]

Ans. $u(o, y) = 0 \,; x = 0$

$$u\,(l, y) = 0 \,; x = l$$

$$u\,(x, y) = 0 \,; y \to \infty$$

$$u\,(x, 0) = f\,(x) : y = 0$$

27. Classify the differential equation. [CO3 - L2]

$$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \, \partial y} + 6\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = 0$$

$$\text{Ans.} A = 3, B = 4, C = 6$$

$$\Delta = B^2 - 4AC = 16 - 72$$

$$= -56 < 0$$

$$\therefore \text{The p.d.e is elliptic.}$$

28. A rod of 5 cm long with insulated sides has its ends A and B kept at 20^0C and 70^0C respectively. Find the steady state temperature distribution of the rod. [CO3 - H2- Apr/May 2015]

Ans. $U = x + 20$

29. Classify the pde $(1+x^2)(4+x^2)u_{xx} + (5+2x^2)u_{xy} + u_{yy} = 0$ [CO3 – L2]

Ans.
$$A = (1+x^2)(4+x^2)$$
$$B = (5+2X^2)$$
$$C = 1$$
$$B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$$
$$= 4 > 0$$

$$\therefore \text{It is hyperbolic.}$$

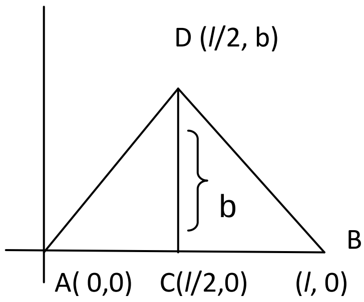
PART B

Problem 1. A Tightly stretched sting of length of 2 l is fastened at both ends the midpoint of the string is displaced by a distance 'b' transversely and the string is released from rest in this position find expression for the transverse displacement of the string at any time during the subsequent motion . [CO3 - H2]

Solution:-

Let 2*l* = L

The equation the line AD



(0 , 0) (L/2, b)

x_1 y_1 x_2 y_2

$$\frac{x-0}{0-L/2} = \frac{y-0}{0-b}$$
$$\frac{x}{-L/2} = \frac{y}{-b}$$
$$\frac{2x}{L} = \frac{y}{b}$$
$$\frac{2bx}{L} = y$$

$$\boxed{\therefore y = \frac{2bx}{L}, \left(0, \frac{L}{2}\right)}$$

The equation of the line DB

(L/2, b) (L, 0)

X_1, y_1 x_2 y_2

$$\frac{x-\frac{L}{2}}{\frac{L}{2}-L} = \frac{y-b}{b-0}$$
$$\frac{x-\frac{L}{2}}{\frac{L}{2}} = \frac{y-b}{b}$$

$$\frac{2x-{L}/{2}}{-{L}/{2}} = \frac{y-b}{b}$$
$$\frac{2x-L}{-L} = \frac{y-b}{b}$$
$$\frac{L-2x}{L} = \frac{y-b}{b}$$
$$\frac{Lb-2xb}{L} = y-b$$
$$\frac{Lb-2xb}{L} + b = y$$
$$\frac{2Lb-2xb}{L} = y$$
$$\frac{2b}{L}(L-x) = y$$

$$\boxed{\therefore y = \frac{2b}{L}(L-x)\ ({L}/{2}, L)}$$

$$f(x) = \begin{cases} \frac{2bx}{L}, & (0, {L}/{2}) \\ \frac{2b}{L}(L-x) & ({L}/{2}, L) \end{cases}$$

The wave equation is

$$\frac{\partial 2y}{\partial t^2} = a^2 \frac{\partial 2y}{\partial x^2}$$

The boundary conditions are

i. y (0,t) = 0, t > 0
ii. y (L,t) = 0, t > 0
iii. $\frac{\partial y}{\partial t}(x,0) = 0,$ (0,L)

iv. $y(x,o) = \begin{cases} \frac{2bx}{L}, & (0, {L}/{2}) \\ \frac{2b}{L}(L-x), & ({L}/{2}, L) \end{cases}$

The correct solution is

y(x,t) = $(A\cos px + B\sin px)(C\cos pat + D\sin pat)$ ⟶ (1)

Apply cond (i) in equation (1)

y(0,t) = (A +0) $(C\cos pat + D\sin pat)$ = 0

A $(C\cos pat + D\sin pat)$ = 0

$C\cos pat + D\sin pat \neq 0$

$\boxed{A = 0}$ sub in equation (1)

(1) ⟹ y(x,t) = $B\sin px\,(C\cos pat + D\sin pat)$ ⟶ (2)

Apply cond x (ii) in equation (2)

y(L,t) = $B\sin pL(C\cos pat + D\sin pat)$ = 0

⟹ B $\neq 0$, $C\cos pat + D\sin pat \neq 0$

$$sinpL = 0$$

$$sinpL = sinn\pi$$

$$pL = n\pi$$

$\boxed{P = {n\pi}/{L}}$ sub in equation (2)

(2) $\Rightarrow$ $\boxed{y(x,t) = Bsin\frac{n\pi^{x}}{L}(C\ cos\frac{p\pi^{at}}{L} + Dsin\ \frac{n\pi^{at}}{L})}$ $\longrightarrow$ (3)

Diff w, r, to 't'

$$\frac{\partial y}{\partial t}(x,t) = Bsin\frac{n\pi x}{L}\left(-C\ sin\ \frac{n\pi at}{L}\ \frac{n\pi a}{L} + Dcos\frac{n\pi at}{L}.\frac{n\pi a}{L}\right) \longrightarrow \quad (4)$$

Apply cond (iii) in (4)

$$\frac{\partial y}{\partial t}(x,0) = Bsin\frac{n\pi x}{L}\left(Dcos\frac{n\pi at}{L}.\frac{n\pi a}{L}\right) = 0$$

$$BDsin\frac{n\pi x}{L}\ \frac{n\pi a}{L} = 0$$

$$B \neq 0, sin\frac{n\pi x}{L} \neq 0\ \frac{n\pi x}{L} \neq 0$$

$\boxed{D = 0}$ sub in (3)

(3) $\Rightarrow$ $y(x,t) = Bsin\frac{n\pi x}{L}\left(C\ cos\frac{n\pi at}{L}\right)$

$= BCsin\frac{n\pi x}{L}\left(C\ cos\frac{n\pi at}{L}\right)$

$= bn\ sin\frac{n\pi x}{L}\left(C\ cos\frac{n\pi at}{L}\right)$

$\boxed{y(x,t) = \sum_{n=1}^{\infty} bn\ \ sin\frac{n\pi x}{L}\ C\ cos\frac{n\pi at}{L}}$ $\longrightarrow$ (5)

apply con of (iv) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} bn\ \ sin\frac{n\pi x}{L} = \begin{cases} \frac{2bx}{L},\ (0, {L}/{2}) \\ \frac{2b}{L}(L-x)\ ({L}/{2}, L) \end{cases}$$

To find of bn , HRSS in (0,L)

$$bn = \frac{2}{L}\int_{O}^{L} f(x)\ \ sin\frac{n\pi x}{L}\ \ dx$$

$$= \frac{2}{L}\ \frac{2b}{L}\left\{\int_{0}^{L/2} x\,sin\frac{n\pi x}{L}\ \ dx + \int_{L/2}^{L}(L-x)sin\frac{n\pi x}{L}\ \ dx\right\}$$

$$\frac{4b}{L^2}\left\{\left[(x)\left(\frac{-\cos\frac{n\pi x}{L}}{n\pi/L}\right)-\left(\frac{-\sin\frac{n\pi x}{L}}{n2\pi^2/L^2}\right)\right]_0^{L/2}\left[(L-x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{L}}{n2\pi^2/L^2}\right)\right]_{L/2}^{L}\right\}$$

$$=\frac{4b}{L^2}\left[\frac{-L}{n\pi}.\frac{L}{2}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{L}{n\pi}.\frac{L}{2}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\right]$$

$$=\frac{4b}{L^2}\left[2\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$\boxed{bn = \frac{8b}{n^2\pi^2}\ \sin\frac{n\pi}{2}}$ sub in Equation (5)

$$y(x,t) = \sum_{n=1}^{\infty}\left(\frac{8b}{n^2\,\pi^2}\right)\sin\frac{n\pi}{2}\ \ \sin\frac{n\pi x}{L}\ \cos\ \frac{n\pi x}{L}$$

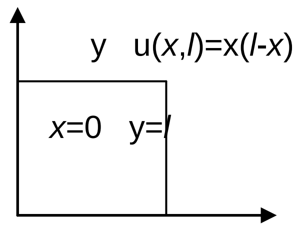
put L = 2*l*

$$\boxed{y(x,t) = \sum_{n=1}^{\infty}\left(\frac{8b}{n^2\,\pi^2}\right)\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2L}\ \cos\ \frac{n\pi at}{2L}}$$

Problem 2. A square plate is bounded by the lines x = 0, y=0, x=20 and y=20 Its faces are insulated the temperature along the upper horizontal edge is given by u (x,20) = x(20-x) when 0<x<20 while the other three edges are kept at 0ºC Find the steady state temperature in the plate. [CO3 - H2-Nov/Dec 2016]

Solution:-

Let us take the sides of the plate be *l* =20 then u (x,y)



Satisfies the Laplace equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$

x=l

The boundary conditions are

x

i) u (o,y) = 0 for 0 < x < *l* *y=0*

ii) u (*l, y*) = 0 for 0 < y < *l*

iii) u (x,o) = 0 for 0 < x < *l*

iv) u (x,*l*) = $x(l-x)$ for 0 < x < *l*

The suitable solution is

u(x,y) = (Acospx + Bsinpx) (C$e^{py}+De^{-py}$) $\longrightarrow$ (1)

Applying (i) in (1)

u(o,y) = A (C$e^{py}+De^{-py}$)

O = A (C$e^{py}+De^{-py}$)

A = O $Ce^{py} + De^{-py} \neq 0$

Put A = 0 in (1)

u (x,y) = Bsinp*x* ($Ce^{py} + De^{-py}$) $\longrightarrow$ (2)

Applying (ii) in *(l)*

u (*l*,y) = Bsinp*l* ($Ce^{py} + De^{-py}$)

0 = Bsinp*l* ($Ce^{py} + De^{-py}$)

B ≠ 0, $Ce^{py} + De^{-py} \neq 0$

∴ sinp*l* = 0

sinp*l* = *sinnπ*

$\boxed{pl = n\pi}$

$p = \frac{n\pi}{l}$

put $p = \frac{n\pi}{l}$ *in (2)*

u(x,y) = Bsin $\left(\frac{n\pi x}{l}\right)$ $\left[Ce^{\frac{n\pi y}{l}} + De^{-\frac{-n\pi y}{l}}\right]$ $\longrightarrow$ (3)

Applying (iii) (3)

u(x,0) = Bsin $\left(\frac{n\pi x}{l}\right)$ [C + D]

0 = Bsin $\left(\frac{n\pi x}{l}\right)$ [C + D]

Bsin $\left(\frac{n\pi x}{l}\right) \neq 0$

∴ *C + D = 0*

$\boxed{D = -C}$

Put $\boxed{D = -C}$ in (3)

u(x,y) = Bsin $\left(\frac{n\pi x}{l}\right)$ $\left[Ce^{\frac{n\pi y}{l}} - Ce^{-\frac{-n\pi y}{l}}\right]$

u(x,y) = BCsin $\left(\frac{n\pi x}{l}\right)$ $\left[e^{\frac{n\pi y}{l}} - e^{o-\frac{-n\pi y}{l}}\right]$ $\boxed{e^x - e^{-x} = 2\,sinhx}$

u(x,y) = BCsin $\left(\frac{n\pi x}{l}\right)$ 2 sin h $\left(\frac{n\pi y}{l}\right)$

u(x,y) = $b_n$sin $\left(\frac{n\pi x}{l}\right)$ sin h $\left(\frac{n\pi y}{l}\right)$ here 2BC = b_n

The most general solution is

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \ \sin h\left(\frac{n\pi y}{l}\right)} \qquad (4)$$

Applying (iv) in (4)

$$u(x,l) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \ \sin h\left(\frac{n\pi l}{l}\right)$$

$$x(l-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \sin hn\pi \qquad (5)$$

This is half range sine series in (O,*l*)

$$x(l-x) = \sum_{n=1}^{\infty} b_n \sin hn\pi \ \sin\left(\frac{n\pi x}{l}\right)$$

To find b_n

$$\text{bn}\ (\sin hn\pi) = \frac{2}{l}\int_0^l f(x)\, sin\left(\frac{n\pi x}{l}\right) dx$$

$$\text{bn}\ (\sin hn\pi) = \frac{2}{l}\int_0^l (lx - x^2)\ sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left[((lx-x^2)\left(-\cos\frac{\frac{n\pi x}{l}}{n\pi/l}\right) - (l-2x)\left(-sin\frac{\frac{n\pi x}{l}}{\frac{n_2\pi^2}{l^2}}\right) + (-2)\left(cos\frac{\frac{n\pi x}{l}}{\frac{n_3\pi^3}{l^3}}\right)\right]_o^l$$

$$\text{bn}\ (\sin hn\pi) = \frac{2}{l}\left[-2\frac{l^3}{n^3\pi^3}\ cosn\pi + 2\ \frac{l^3}{n^3\pi^3}\right]$$

$$= \frac{2}{l}\,.\,2\frac{l^3}{n^3\pi^3}[cosn\pi + 1]$$

$$= \frac{4l^2}{n^3\pi^3}\ [-(-1)^n + 1]$$

$$= \frac{4l^2}{n^3\pi^3}\ [1-(-1)^n]$$

When,

n is even $\Longrightarrow$ $b_n = 0$

n is odd $\Longrightarrow$ $b_n (\sinh n\pi) = \frac{8l^2}{n^3\pi^3}$

$$b_n = \frac{8l^2}{n^3\pi^3}\cdot\frac{1}{\sinh n\pi}$$

$$\boxed{b_n = \frac{8l^2}{n^3\pi^3 \sinh n\pi}}$$

sub b_n in equation (4)

$$u(x,y) = \sum_{n=1,3}^{\infty}\left(\frac{8l^2}{n^3\pi^3\sinh n\pi}\right) \sin\frac{n\pi x}{l}\ sinh\frac{n\pi y}{l}$$

put $\boxed{l=20}$

$$u(x,y) = \sum_{n=1,3}^{\infty}\left(\frac{3200}{n^3\pi^3 \sinh n\pi}\right)\sin\frac{n\pi x}{20}\ \sinh\frac{n\pi y}{20}$$

Problem 3. A tightly stretched string with fixed end points $x = 0$ & $x = l$ is initially in a position given by $y(x,0) = y_0 \sin^3\frac{\pi x}{l}$. If it is released from rest from this position. Find the displacement $'y'$at any distance $'x'$ from one end at any time $,t,$. [CO3 - H2- Nov/Dec 2016]

Given: $y(x,0) = y_0 \sin^3\frac{\pi x}{l} = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$

$$\left[\because \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)\right]$$

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

The boundary conditions are,

i) $y(0,t) = 0, t > 0$

ii) $y(l,t) = 0, t > 0$

iii) $\frac{\partial y}{\partial t}(x,0) = 0, 0 \le x \le l$

iv) $y(x,0) = f(x) = \left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right], 0 \le x \le l$

The correct solution is

$y(x,t) = (A\cos Px + B\sin Px)\ (C\cos Pat + D\sin Pat) \to (1)$

Apply cond. (i)in eqn. (1)

$(1) \Rightarrow y(0,t) = (A + 0)(C\cos Pat + D\sin Pat) = 0$

$A\ (C\cos Pat + D\sin Pat) = 0$

$\Rightarrow C\cos Pat + D\sin Pat \neq 0$

$\therefore A = 0 \text{ sub. in } (1)$

$y(x,t) = B\sin Px\,(C\cos Pat + D\sin Pat) \rightarrow (2)$

Apply cond. (ii)in eqn. (2)

$y(x,t) = B\sin Pl\,(C\cos Pat + D\sin Pat) = 0$

$\Rightarrow B\sin Pl\,(C\cos Pat + D\sin Pat) = 0$

$\Rightarrow C\cos Pat + D\sin Pat \neq 0, B \neq 0$

$\therefore \sin Pl = 0 = \sin n\pi$

$\Rightarrow Pl = n\pi$

$P = \frac{n\pi}{l} \text{sub. in } (2)$

$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} = D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$

Diff. w, r to 't' in (3),

$\frac{\partial y}{\partial t}(x,t) = B\sin\frac{n\pi x}{l}\left[\left(-C\sin\frac{n\pi at}{l}.\frac{n\pi a}{l}\right) + \left(D\cos\frac{n\pi at}{l}.\frac{n\pi a}{l}\right)\right] \rightarrow (4)$

Apply cond. (iii)in eqn. (4)

$(4) \Rightarrow \frac{\partial y}{\partial t}(x,0) = B\sin\frac{n\pi x}{l}\left(0 + D\ \frac{n\pi a}{l}\right) = 0$

$B\sin\frac{n\pi x}{l}\left(D\ \frac{n\pi a}{l}\right) = 0$

$\Rightarrow B \neq 0, \frac{n\pi a}{l} \neq 0, \sin\frac{n\pi x}{l} \neq 0$

$\therefore D = 0 \text{ sub. in } (3)$

$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l}\right) = BC\ \sin\frac{n\pi x}{l}.\cos\frac{n\pi at}{l}$

$= b_n\sin\frac{n\pi x}{l}.\cos\frac{n\pi at}{l}$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi at}{l}$$

$$y(x,t) = b_1 \sin\frac{\pi x}{l} \cdot \cos\frac{\pi at}{l} + b_2 \sin\frac{2\pi x}{l} \cdot \cos\frac{2\pi at}{l} + \cdots (5)$$

Apply cond. (iv)in eqn. (5)

$$y(x,0) = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \cdots = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$$b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

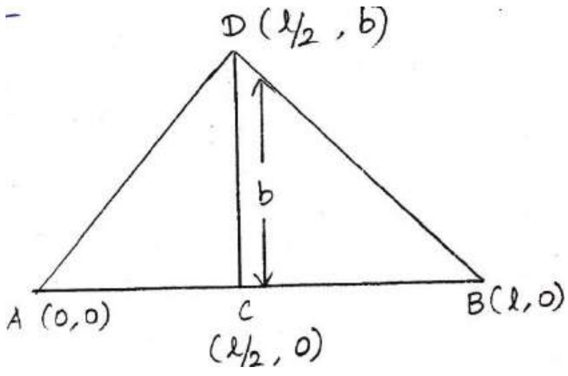
Equating coefficients on both the sides,

$$b_1 = \frac{3y_0}{4}, b_2 = 0, b_3 = -\frac{y_0}{4} \text{sub. in } (5)$$

$$(5) \Rightarrow y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l} \cdot \cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} \cdot \cos\frac{3\pi at}{l}.$$

Problem 4. A string is tightly stretched & its ends are fastened at two points *x*=0 & *x*=*l*.The midpoint of the string is displaced transversely through a small distance 'b' & the string is released from rest in that position. Find an expression for the transverse displacement of the string at any time during the motion. [CO3 - H2]

Given:



The eqn. of the line AD

$(0,0)(l/2, b)$
$x_1\ y_1 \quad x_2\ y_2$

$$\frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2} \Rightarrow \frac{x - 0}{0 - l/2} = \frac{y - 0}{0 - b}$$

$$\frac{x}{-l/2} = \frac{y}{-b}$$

$$\frac{2x}{l} = \frac{y}{b} \Rightarrow y = \frac{2bx}{l} = f(x)$$

$$\therefore f(x) = \frac{2bx}{l}, (0, l/2)$$

The eqn. of the line DB is $\underset{x_1\, y_1\ \ x_2\, y_2}{(l/2,b)(l,0)}$

$$\frac{x-l/2}{l/2-l}=\frac{y-b}{b}\Rightarrow\frac{\frac{2x-l}{\cancel{2}}}{-l/\cancel{2}}=\frac{y-b}{b}$$

$$\Rightarrow\frac{l-2x}{l}=\frac{y-b}{b}\ \Rightarrow\frac{lb-2bx}{l}=y-b$$

$$\Rightarrow y=\frac{lb-2bx}{l}+b\Rightarrow y=\frac{2lb-2bx}{l}$$

$$\Rightarrow\frac{2b}{l}(l-x)=y=f(x)$$

$$\therefore f(x)=\begin{cases}2bx/l,(0,l/2)\\2b/l(l-x),(l/2\,,l)\end{cases}$$

The wave equation is $\frac{\partial^2y}{\partial t^2}=a^2\frac{\partial^2y}{\partial x^2}$

The boundary conditions are,

$i)\ y(0,t)=0,t>0$

$ii)\ y(l,t)=0,t>0$

$iii)\frac{\partial y}{\partial t}(x,0)=0,0\le x\le l$

$iv) y(x,0)=f(x)=\begin{cases}2bx/l,(0,l/2)\\2b/l(l-x),(l/2\,,l)\end{cases},0\le x\le l$

The correct solution is

$y(x,t)=(A\cos Px+B\sin Px)\ (C\cos Pat+D\sin Pat)\to(1)$

Apply cond. (i)in eqn. (1)

$(1)\Rightarrow y(0,t)=A(C\cos Pat+D\sin Pat)=0$

$\Rightarrow C\cos Pat+D\sin Pat\neq 0$

$\therefore A=0\ sub.in\ (1)$

$y(x,t)=B\sin Px\ (C\cos Pat+D\sin Pat)\to(2)$

Apply cond. (ii)in eqn. (2)

$$y(x,t) = B\sin Pl\,(C\cos Pat + D\sin Pat) = 0$$

$$\Rightarrow C\cos Pat + D\sin Pat \neq 0, B \neq 0$$

$$\therefore \sin Pl = 0 = \sin n\pi$$

$$\Rightarrow Pl = n\pi$$

$$P = \frac{n\pi}{l} sub.in\ (2)$$

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\,cos\frac{n\pi at}{l} + D\,sin\frac{n\pi at}{l}\right) \to (3)$$

Diff. w,r to $'t'$ in (3),

$$\frac{\partial y}{\partial t}(x,t) = B\sin\frac{n\pi x}{l}\left[\left(-C\,sin\frac{n\pi at}{l}.\frac{n\pi a}{l}\right) + \left(D\,cos\frac{n\pi at}{l}.\frac{n\pi a}{l}\right)\right] \to (4)$$

Apply cond. (iii)in eqn. (4)

$$(4)\Rightarrow\frac{\partial y}{\partial t}(x,0) = B\sin\frac{n\pi x}{l}\left(0 + D\ \frac{n\pi a}{l}\right) = 0$$

$$B\sin\frac{n\pi x}{l}\left(D\ \frac{n\pi a}{l}\right) = 0$$

$$\Rightarrow B \neq 0, \frac{n\pi a}{l} \neq 0, \sin\frac{n\pi x}{l} \neq 0$$

$$\therefore D = 0\ sub.in\ (3)$$

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\,cos\frac{n\pi at}{l}\right) = BC\ \ sin\frac{n\pi x}{l}.cos\frac{n\pi at}{l}$$

$$= b_n\sin\frac{n\pi x}{l}.cos\frac{n\pi at}{l}$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}.cos\frac{n\pi at}{l} \to (5)$$

Apply cond. (iv)in eqn. (5)

$$y(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}=\begin{cases} 2bx/l, (0, l/2) \\ 2b/l(l-x), (l/2\,, l) \end{cases}$$

To find $b_{n,}$ HRSS in $(0,l)$,

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx=\frac{2}{l}\left[\frac{2b}{l}\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}dx+\frac{2b}{l}\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$=\frac{2}{l}\,.\frac{2b}{l}\left\{\left[(x)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right)-\left(-\frac{\sin n\pi x/l}{n^2\pi^2/l^2}\right)\right]_0^{\frac{l}{2}}\right.$$

$$+\left[(l-x)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right)-(-1)\left(-\frac{\sin n\pi x/l}{n^2\pi^2/l^2}\right)\right]_{\frac{l}{2}}^{l}$$

$$=\frac{4b}{l^2}\left[-\frac{l}{n\pi}.\frac{l}{2}\cos\frac{n\pi}{l}.\frac{l}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{l}.\frac{l}{2}+\frac{l}{n\pi}.\frac{l}{2}\cos\frac{n\pi}{l}.\frac{l}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{l}.\frac{l}{2}\right]$$

$$=\frac{4b}{l^2}\left[2\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]=\frac{8b}{n^2\pi^2}\ \sin\frac{n\pi}{2}$$

$$\Rightarrow b_n=\ \frac{8b}{n^2\pi^2}\ \sin\frac{n\pi}{2}\ sub.in\ (5),$$

$$(5)\Rightarrow y(x,t)=\sum_{n=1}^{\infty}\left(\frac{8b}{n^2\pi^2}\ \sin\frac{n\pi}{2}\right).\sin\frac{n\pi x}{l}.\cos\frac{n\pi at}{l}$$

Problem 5. A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially at rest in its position. If it is set vibrating givingeach point at a velocity $\lambda x(l-x)$. Find the displacement at any point at any time $'t'$. [CO3 - H2-Nov/Dec 2014]

Given: $f(x)=\lambda x(l-x)=\lambda(lx-x^2)$

The wave equation is

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are,

$i)\ y(0,t)=0, t>0$

$ii)\ y(l,t) = 0, t > 0$

$iii)\ y(x,0) = 0, 0 \le x \le l$

$iv)\ \dfrac{\partial y}{\partial t}(x,0) = f(x) = \lambda(lx - x^2), 0 \le x \le l$

The correct solution is

$y(x,t) = (A\cos Px + B\sin Px)\ (C\cos Pat + D\sin Pat) \to (1)$

Apply cond. (i)in eqn. (1)

$(1) \Rightarrow y(0,t) = A(C\cos Pat + D\sin Pat)$

$\Rightarrow C\cos Pat + D\sin Pat \neq 0$

$\therefore A = 0\ sub.in\ (1)$

$y(x,t) = B\sin Px\ (C\cos Pat + D\sin Pat) \to (2)$

Apply cond. (ii)in eqn. (2)

$y(l,t) = B\sin Pl\ (C\cos Pat + D\sin Pat) = 0$

$\Rightarrow B\sin Pl\ (C\cos Pat + D\sin Pat) = 0$

$\Rightarrow C\cos Pat + D\sin Pat \neq 0, B \neq 0$

$\therefore \sin Pl = 0 = \sin n\pi$

$\Rightarrow Pl = n\pi$

$P = \dfrac{n\pi}{l} sub.in\ (2)$

$(2) \Rightarrow y(x,t) = B\sin\dfrac{n\pi x}{l}(C\cos Pat + D\sin Pat) \to (3)$

Apply cond. (iii)in eqn. (3)

$y\ (x,0) = B\sin\dfrac{n\pi x}{l}(c + 0) = 0$

$\Rightarrow B\sin\dfrac{n\pi x}{l}.c = 0$

$$\Rightarrow B \neq 0, \sin\frac{n\pi x}{l} \neq 0, \qquad \therefore C = 0\ sub.in\ (3)$$

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(D\sin\frac{n\pi at}{l}\right)$$

$$y(x,t) = BD\sin\frac{n\pi x}{l}.\sin\frac{n\pi at}{l}$$

$$y(x,t) = b_n\sin\frac{n\pi x}{l}.\sin\frac{n\pi at}{l}$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}.cos\frac{n\pi at}{l} \to (4)$$

Diff. $w,r\ to\ 't'$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\ .cos\frac{n\pi at}{l}\ \left(\frac{n\pi a}{l}\right) \to (5)$$

Apply cond. (iv)in eqn. (5)

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}.\frac{n\pi a}{l} = \lambda(lx - x^2)$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}.\frac{n\pi a}{l} = \lambda(\frac{ln}{x} - 2\frac{n^{2^2}}{x^2}\ )$$

To find $b_{n,}$ HRSS in $(0,l)$,

$$b_n = \frac{2\lambda}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2\lambda}{l}\ (lx - x^2)\left[\left(-\frac{\cos n\pi x/l}{n\pi/l}\right) - (l - 2x)\left(-\frac{\sin n\pi x/l}{n^2\pi^2/l^2}\right) + (-2)\left(-\frac{\cos n\pi x/l}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\ .\frac{2l^3}{n^3\pi^3}[1 - (-1)^n] = \frac{4\lambda l^2}{n^3\pi^3}\ [1 - (-1)^n]$$

When $'n'$ is even, $b_n = 0$

$'n'$ is odd, $b_n = \frac{8\lambda l^2}{n^3\pi^3}$

$b_n \cdot \frac{n\pi a}{l} = \frac{8\lambda l^2}{n^3\pi^3} \Rightarrow b_n = \frac{8\lambda l^3}{n^4\pi^4 a}\ sub.in\ (4).$

$$(4) \Rightarrow y(x,t) = \sum_{n=1,3}^{\infty} \left(\frac{8\lambda l^3}{n^4\pi^4 a}\right). \sin\frac{n\pi x}{l}. \sin\frac{n\pi at}{l}$$

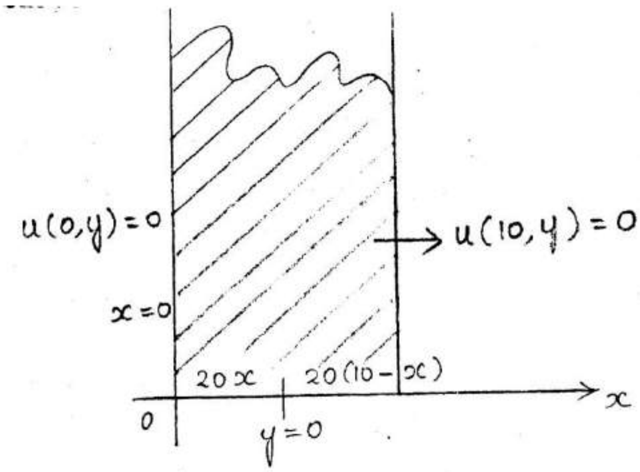
Problem 6. A rectangular plate with insulated surface is 10 cm wide and compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge y=0 is given by

$$\boldsymbol{u = \begin{cases} 20x, 0 \le x \le 5 \\ 20(10-x), 5 \le x \le 10 \end{cases}}$$

And all the other three edges are kept at 0º c. Find the steady state temperature at any point in the plate. [CO3 - H2-May/Jun 2013]

Solution:

Let $10 = l$



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The boundary conditions are,

$i)\ u\ (0,y) = 0, for\ all\ y$

$ii)\ u\ (l,y) = 0, for\ all\ y$

$iii)\ u\ (x,\infty) = 0, 0 \le x \le l$

$$iv)\ u\ (x,0) = \begin{cases} 20x, 0 \le x \le l/2 \\ 20(10-x), l/2 \le x \le l \end{cases}$$

The correct solution is

$u\ (x,y) = (A\cos Px + B\sin Px)\ (Ce^{py} + De^{-py}) \rightarrow (1)$

Apply cond. (i)in eqn. (1)

$u\ (0,y) = A(Ce^{py} + De^{-py}) = 0$

$\Rightarrow Ce^{py} + De^{-py} \neq 0$

$\therefore A = 0\ sub.in\ (1)$

$(1) \Rightarrow u\ (x, y)$

$u(x, y) = B \sin Px\ \ (Ce^{py} + De^{-py}) \rightarrow (2)$

Apply cond. (ii)in eqn. (2)

$u\ (l, y) = B \sin Pl\ \ (Ce^{py} + De^{-py})$

$0 = B \sin Pl\ \ (Ce^{py} + De^{-py})$

$\Rightarrow B \neq 0, Ce^{py} + De^{-py} \neq 0$

$\therefore \sin Pl = 0 = \sin n\pi$

$\Rightarrow Pl = n\pi$

$\Rightarrow P = \frac{n\pi}{l} sub.in\ (2)$

$u\ (x, y) = B \sin \frac{n\pi x}{l} \left(Ce^{n\pi y/l} + De^{-n\pi y/l}\right) \rightarrow (3)$

Apply cond. (iii)in eqn. (3)

$u\ (x, \infty) = B \sin \frac{n\pi x}{l} (Ce^{\infty} + De^{-\infty}) = 0$

$\Rightarrow B \sin \frac{n\pi x}{l} . C = 0$

$\Rightarrow B \neq 0, \qquad \sin \frac{n\pi x}{l} \neq 0$

$\therefore C = 0\ sub.in\ (3)$

$u\ (x, y) = B \sin \frac{n\pi x}{l} \left(De^{-n\pi y/l}\right)$

$= BD \sin \frac{n\pi x}{l} . e^{-n\pi y/l} = b_n \frac{n\pi x}{l}\ e^{-n\pi y/l}$

$u\ (x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . e^{-n\pi y/l} \rightarrow (4)$

Apply cond. (iv)in eqn. (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \begin{cases} 20x, (0, l/2) \\ 20(l-x), (l/2\,, l) \end{cases}$$

To find b_n, HRSS in $(0, l)$,

$$b_n = \frac{2}{l}\int_0^l f(x)\ \sin\frac{n\pi x}{l}\ dx$$

$$= \frac{2}{l}\ .20\left[\int_0^l x\sin\frac{n\pi x}{l}\ dx + \int_{\frac{l}{2}}^l (l-2x)\sin\frac{n\pi x}{l}\ dx\right]$$

$$= \frac{40}{l}\ (x)\left[\left(-\frac{\cos n\pi x/l}{n\pi/l}\right) - \left(-\frac{\sin n\pi x/l}{n^2\pi^2/l^2}\right)\right]_0^{\frac{l}{2}} + \left[(l-x)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right) - (-1)\left(-\frac{\sin n\pi x/l}{n^2\pi^2/l^2}\right)\right]_{\frac{l}{2}}^{l}$$

$$= \frac{40}{l}\left[-\frac{l}{n\pi}\ .\frac{l}{2}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l}{n\pi}\ .\frac{l}{2}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{40}{l}\ .\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \frac{80l}{n^2\pi^2}\sin\frac{n\pi}{2}\ sub.in\ (4)$$

$$u\ (x,y) = \sum_{n=1}^{\infty}\frac{80l}{n^2\pi^2}\sin\frac{n\pi}{2}\ \sin\frac{n\pi x}{l}.e^{-n\pi y/l}$$

Put $l = 10$

$u\ (x,y) = \sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\ \sin\frac{n\pi x}{10}.e^{-n\pi y/10}.$

UNIT – IV FOURIER TRANSFORMS

PART –A

1. State the inversion formula for a Fourier Transform. [CO4 – L1]

Solution:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{-isx}\, ds}$$

2. State the convolution theorem of Fourier Transform. [CO4 – L1]

Solution:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{-isx}\, ds}$$

3. Find Fourier cosine Transform of e^{-ax}. [CO4 - H2]

Solution:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{-isx}\, ds}$$

$$F_C[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-ax}\cos sx\ dx$$

$$\boxed{F_C[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]}$$

4. Prove that $F[f(x-a)] = e^{ios}F(s)$. [CO4 - H2-Apr/May 2015]

Solution:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{-isx}\, ds}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\, dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{-isx}\, dx$$

Put $x - a = t \quad \Rightarrow \quad x = t + a$

$\Rightarrow dx = dt$

$$\therefore F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-is(t+a)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}\, e^{-isa}dt$$

$$= \frac{e^{-isa}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}\, dt$$

$$\boxed{F[f(x-a)] = e^{isa}F(s)}$$. Hence proved

5. Find the Fourier sine transform of $\frac{1}{x}$. **[CO4 - H2 – Apr/May 2015]**

Solution: $F_S[1/x] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \frac{\pi}{2} \quad \left[\because \int_0^\infty \frac{\sin sx}{x} dx = \pi/2\right]$$

$$\boxed{\therefore \quad F_S[1/x] = \sqrt{\pi/2}}$$

6. State the Fourier Integral theorem. **[CO4 – L1-May/June 2016]**

Solution:

The Fourier Integral theorem of $f(x)$ in the interval $(-l, l)$ is

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) cos\lambda(t-x) dx \, d\lambda.}$$

7. Write down the Fourier transform pair. **[CO4 – L1]**

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx} \, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F[f(x)] e^{-isx} \, ds$$

8. Find the Fourier sine transform of e^{-x}. **[CO4 - H2]**

Solution:

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_S[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$$\boxed{\therefore F_S[e^{-x}] = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+1}\right)}$$

9. Write down the Fourier sine transform pair . **[CO4 – L1]**

Solution: $F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S[f(x)] \sin sx \, ds$$

10. Write down the Fourier cosine transform pair of formulae. [CO4 – L1]

Solution: $F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\ dx$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_C[f(x)]\cos sx\ ds$$

11. Prove that $F[e^{iax}f(x)] = F(s+a)$, where $F[f(x)] = F(s)$. [CO4 - H2]

Solution: $F\left[e^{iax}f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}\, f(x)e^{isx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}dx$$

$$\boxed{F\left[e^{iax}f(x)\right] = F(s+a)}$$

12. Prove that $F[f(ax)] = \frac{1}{a}F(s/a), a > 0$. [CO4 - H2- Nov/Dec 2015]

Solution: $F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}dx$

Put $ax = t \quad \Rightarrow \quad x = t/a$

$adx = dt \quad \Rightarrow \quad dx = dt/a$

$$\therefore\ F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{is(t/a)}\,\frac{dt}{a}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i(S/a)t}dt$$

$$\boxed{F[f(ax)] = \frac{1}{a}F(S/a)}$$

13. Find Fourier cosine transform of e^{-x} . [CO4 - H2]

Solution: $F_C[e^{-x}] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos sx\, dx$

$$\boxed{F_C[e^{-x}] = \sqrt{\frac{2}{\pi}}\left(\frac{1}{s^2+1}\right)}$$

14. Find (a) $F[x^n f(x)]$ (b) $F\left[\frac{d^n f(x)}{dx^n}\right]$ interms of Fourier transform of $f(x)$. [CO4 - H2]

(a) $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n}F(s)$

(b) $F\left[\frac{d^n}{dx^n}f(x)\right] = (-is)^n F(s)$

(c)

15. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$ [CO4 - H2]

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i\, sin\, sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx = \frac{1}{\sqrt{2\pi}} 2. \int_{-a}^{a} \cos sx\, dx$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s}\right]$$

16. If F(s) us the Fourier transform of $f(x)$. Write the formula for the Fourier transform of $f(x) \cos ax$ in terms of F. [CO4 – L1]

Solution:

$$F[f(x) \cos ax] = {}^1/_2\, [F(s+a) + F(s-a)]$$

17. **If $F_S(s)$ is the Fourier sine Transform of $f(x)$, show that $F_S[f(x) \cos ax] = \frac{1}{2}[F_s(s+a) + F_S(s-a)]$. [CO4 - H2]**

Solution: $F_S[f(x) \cos ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos ax \sin sx\, dx$

$$= \frac{1}{2}. \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(s+a)x + \sin(s-a)\, x]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(s+a)x\, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(s-a)\, x\, dx\right]$$

$$\boxed{F_S[f(x) cos\ ax] = \frac{1}{2}[F_S(s+a) + F_S(s-a)]}$$

18. Solve the integral equation $\int_0^\infty f(x) \cos \lambda x\ dx = e^{-\lambda}$. [CO4 - H2]

Solution: Given: $\int_0^\infty f(x) \cos \lambda x\, dx = e^{-\lambda}$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x\, dx = \sqrt{\frac{2}{\pi}} e^{-\lambda}$$

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}} e^{-\lambda}$$

$$\therefore f(x) = F_C^{-1}\left[\sqrt{\frac{2}{\pi}} e^{-\lambda}\right] = \sqrt{\frac{2}{\pi}}\ F_C^{-1}[e^{-\lambda}]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\lambda} \cos \lambda x\, dx$$

$$f(x) = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right).$$

19. State Parseval's Identity theorem in Fourier Transform. [CO4 – L1-May/June 2014]

Solution:

$$\boxed{\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |F(x)|^2 dx}$$

20. Find the Fourier sine transform of $f(x) = 1$ is $(0, l)$. [CO4 - H2]

Solution: $F[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,sin\frac{n\pi x}{l}\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^l sin\frac{n\pi x}{l}\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{-cosn\pi x/l}{n\pi/l}\right]_0^l$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{l}{n\pi}[cosn\pi - cos0]$$

$$F[f(x)] = -\frac{l}{n\pi}\sqrt{\frac{2}{\pi}}[(-1)^n - 1]$$

21. Find the Fourier transform of $e^{-\propto|x|}, \propto > 0$. [CO4 - H2]

Solution: $F\left[e^{-\propto|x|}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\propto|x|}e^{isx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\propto|x|}\,(\cos sx + i\,sin\,sx)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\propto|x|}\cos sx\,dx + 0$$

$$F\left[e^{-\propto|x|}\right] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-\propto x}\cos sx\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{\propto}{\propto^2+s^2}\right)$$

22. Find the Fourier cosine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$ [CO4 - H2]

Solution: $F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[(x)\frac{\sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^1 + \left[(2-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_1^2\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\cancel{\frac{\sin s}{s}} + \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} \cancel{-\frac{\sin s}{s}} + \frac{\cos s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2\cos s}{s^2} - \frac{(1+\cos 2s)}{s^2}\right]$$

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{2\cos s}{s^2} - \frac{2\cos^2 s}{s^2}\right] = \sqrt{\frac{2}{\pi}}\frac{2\cos s\,(1-\cos s)}{s^2}$$

23. Find the Fourier transform of $f(x) = \begin{cases} e^{iKx}, & a < x < b \\ 0, & x < a, x > b \end{cases}$ [CO4 - H2]

Solution: $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b e^{iKx}\,e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_a^b e^{i(K+s)x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(K+s)x}}{i(K+s)}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{i(K+s)}\left[e^{i(K+s)b} - e^{i(K+s)a}\right]$$

24. State the Fourier transform of the derivatives of a function. [CO4 – L1-May/June 2016]

Solution: $\boxed{F[x^n f(x)] = (-1)^n \frac{d^n}{ds^n}(s)}$

PART B

1. Show that $e^{-\frac{x^2}{2}}$ is self – reciprocal under Fourier cosine transform. [CO4 - H2 – May/June 2016]

Soln:

We know that $F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$F_C[e^{-x^2/2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_{-\infty}^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \text{ R.P} \int_{-\infty}^\infty e^{-x^2/2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P} \int_{-\infty}^\infty e^{-x^2/2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P} \int_{-\infty}^\infty e^{-\frac{-x^2}{2}+isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P} \int_{-\infty}^\infty e^{-\frac{-1}{2}(x^2-2isx)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P} \int_{-\infty}^\infty e^{-\frac{-1}{2}(x^2-2isx+(is)^2-(is)^2)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P} \int_{-\infty}^\infty e^{-\frac{-1}{2}(x-is)^2} \; e^{-s^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P} \int_{-\infty}^\infty e^{-[\frac{x-is}{\sqrt{2}}]^2} \; e^{-s^2/2} \, dx$$

Put $y = \frac{x-is}{\sqrt{2}}$ | $x \to -\infty \implies y \to -\infty$

$dy = \frac{dx}{\sqrt{2}}$ | $x \to \infty \implies y \to \infty$

$$\therefore F_C\left[e^{-x^2/2}\right] = \frac{1}{\sqrt{2\pi}} \text{ R.P} \int_{-\infty}^\infty e^{-y^2} e^{-s^2/2} \; \sqrt{2} \, dy$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \text{ R.P} \int_{-\infty}^\infty e^{-y^2} \, dy$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \sqrt{\pi} \qquad [\because \int_{-\infty}^\infty e^{-y^2} \, dy = \sqrt{\pi}]$$

$$\therefore F_C\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

Hence f (x) =$e^{-s^2/2}$ is self reciprocal with respect to Fourier cosine transform.

2. Find the Fourier sine transform of [CO4 - H2 – Apr/May 2011]

$$f(x) = \begin{cases} x, & 0 < x\ 1 \\ 2-x, & 1 < x\ 2 \\ 0, & x > 2 \end{cases}.$$

Solution:

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin sx\, dx + \int_1^2 (2-x)\sin sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[x\left(\frac{-\cos sx}{s}\right) - (1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^1 + \left[(2-x)\left(\frac{-\cos sx}{s}\right) - (-1)\left(\frac{-\sin sx}{s^2}\right)\right]_1^2\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[\frac{-\cos s}{s} + \frac{\sin s}{s^2}\right] + \left[\frac{-\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right]\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\, sins - \sin 2s}{s^2}\right]$$

3.Using Parseval's identity calculate [CO4 – L3]

i) $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ **if a > 0** **ii)** $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}$ **dx.**

Solution:

$f(x) = e^{-ax}$,

$F_S[f(x)] = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) = F_S(s)$ **…. (1)**

$F_C[f(x)] = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right) = F_C(s)$ **…. (2)**

(i) Parseval's identity for Fourier sine transform is

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_s(s)|^2\, ds$$

Here $f(x) = e^{-ax}$

$$\int_0^\infty |e^{-ax}|^2\, dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right]^2 ds \qquad \text{[Using (1)]}$$

$$\int_0^\infty e^{-2ax}\, dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = \frac{\pi}{2}\int_0^\infty e^{-2ax}\, dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{\pi}{2}\left[0 + \frac{1}{2a}\right] = \frac{\pi}{4a}$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = \frac{\pi}{4a}$$

$$\boxed{\int_0^\infty \frac{x^2}{(s^2+a^2)^2}\, dx = \frac{\pi}{4a}}$$

ii) Using Parsevals identity for Fourier cosine transform.

$$\int_0^\infty |f(x)|^2 \, dx = \int_0^\infty |F_s(s)|^2 \, ds$$

$$\int_0^\infty |e^{-ax}|^2 \, dx = \int_0^\infty \left\{\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right\}^2 ds \qquad \text{[Using (2)]}$$

$$\frac{2}{\pi} \int_0^\infty \frac{a^2}{(s^2+a^2)^2} \, ds = \int_0^\infty (e^{-ax})^2 \, dx = \int_0^\infty e^{-2ax} \, dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a}$$

$$\int_0^\infty \frac{a^2}{(s^2+a^2)^2} \, ds = \frac{\pi}{4a}$$

$$\boxed{\int_0^\infty \frac{dx}{(x^2+a^2)^2} \, ds = \frac{\pi}{4a^3}}$$

4. Find Fourier transform of $e^{-a|x|}$ and hence deduce that **[CO4 - H2]**

i) $\int_0^\infty \frac{\cos xt}{a^2+t^2} \, dt = \frac{\pi}{2a} e^{-a|x|}$ **ii) F $\left[x\, e^{-a|x|}\right] = (i) \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$**

Solution:

Fourier transform of f(x) is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).\, e^{isx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos sx \, dx = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$F\left[e^{-a|x|}\right] = \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right) \qquad \textbf{.... A}$$

Using inversion formula,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} . \left(\frac{a}{s^2+a^2}\right) e^{-isx} \, ds$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \left(\frac{\cos sx - i \sin sx}{s^2+a^2}\right) ds$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \left(\frac{\cos sx}{s^2+a^2}\right) ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2+a^2} \, ds$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos sx}{s^2+a^2} \, ds = \frac{\pi}{2a} f(x) = \frac{\pi}{2a} . e^{-a|x|} \qquad \textbf{.... B}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos tx}{t^2+a^2} \, dt = \frac{\pi}{2a} e^{-a|x|}}$$

ii) $F\left[x\, e^{-a|x|}\right] = (-i) \sqrt{\frac{2}{\pi}} \frac{d}{ds} \left(\frac{a}{s^2+a^2}\right)$ [By property]

$= (-i)\sqrt{\frac{2}{\pi}}\ \left[\frac{0-2as}{(s^2+a^2)^2}\right]$

$$\boxed{F\left[x\, e^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}}\ \frac{2as}{(s^2+a^2)^2}}$$

5. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier transform . [CO4 - H2- Nov/Dec - 2014]

$$f(x) = e^{-ax}, g(x) = e^{-bx}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right), F_c[g(x)] = \sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+b^2}\right)$$

Using properly,

$$\int_0^\infty F_c[f(x)]F_c[g(x)]ds = \int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)\sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+b^2}\right)ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \int_0^\infty e^{-(a+b)x}$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= -\frac{1}{(a+b)}[e^{-\infty} - e^0]$$

$$= \frac{-1}{a+b}[e^{-\infty} - 1] = \frac{1}{a+b}$$

$$\Rightarrow \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a+b} \Rightarrow \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

Put $s = x$

$$\therefore \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

6. show that $e^{-x^2/2}$ is a self reciprocal under Fourier Transform. [CO4 - H2-Nov/Dec 2016]

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\, e^{isx}e^{s^2/2}e^{-s^2/2}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2+isx+s^2/2}\, e^{-s^2/2}dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-is)^2/2}\, e^{-s^2/2}dx$$

Put $u = \frac{x-is}{\sqrt{2}}$ when $\substack{x=-\infty, u=-\infty \\ x=\infty, u=\infty}$;

$du = \frac{dx}{\sqrt{2}}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}\, e^{-s^2/2}\sqrt{2}du$$

$$= e^{-s^2/2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\, du$$

$$= e^{-s^2/2}\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-u^2}\, du$$

$$= e^{-s^2/2}\frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = e^{-s^2/2}$$

$$\left[\because \int_0^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}\right]$$

$$\therefore F[f(x)] = e^{-s^2/2}$$

Hence proved.

7. State & prove convolution theorem. [CO4 - H2 – May/June 2012]

If $F(s)\&G(s)$ are the FT of $f(x)\&\ g(x)$respectively, then the FT of the convolution of $f(x)\&\ g(x)$is the product of their FT.

ie, $F\,[(f*g)(x)] = F(s).G(s)$

proof:

$$F\,[(f*g)(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)g(x-t)\right]dt\, e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x-t)e^{isx}dx\right]dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x-t)e^{isx}e^{ist}e^{-ist}dx\right]dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x-t)e^{is(x-t)}d(x-t)\right]e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{ist}dt\,.\,G(s)$$

$$F\,[(f*g)(x)] = F(s).G(s)$$

Hence Proved.

8. Find FST of $\frac{e^{-ax}}{x}$ [CO4 - H2]

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin sx\,dx$$

$$F(s) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-ax}}{x}\sin sx\,dx$$

Diff. w, r, to ‘s’ on both sides,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\sin sx\,dx\right)$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty x\frac{e^{-ax}}{x}\cos sx\,dx$$

$$\frac{d}{ds}F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\frac{d}{ds}F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$$

Taking integration on both sides w, r, to ‘s’

$$F(s) = \sqrt{\frac{2}{\pi}}\int \frac{a}{s^2+a^2}ds$$

$$[\therefore \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}a\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

9. Find FST of e^{-x} hence deduce that $\int_0^\infty \frac{x \sin mx}{1+X^2} dx = \frac{\pi}{2} e^{-m}$ [CO4 - H2]

$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx \quad \because F_s[e^{-ax}] = \frac{s}{s^2+a^2S}$ Here a=1

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+1}\right]$$

$$\left[\because F_s[f(x)] = F_s[e^{-ax}] = \frac{s}{s^2+a^2}\right]$$

IFST $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx \, ds$$

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+1}\right) \sin sx \, ds$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{1+S^2} ds$$

$$\frac{2}{\pi} e^{-x} = \int_0^\infty \frac{s \sin sx}{s^2+1} ds$$

Put $x = m$

$$\int_0^\infty \frac{s\sin ms}{s^2+1} ds = \frac{2}{\pi} e^{-m}$$

Put $s = x$

$$\int_0^\infty \frac{x \sin mx}{x^2+1} dx = \frac{2}{\pi} e^{-m}$$

10. Find FCT of e^{-x^2} [CO4 - H2-Apr/May - 2014]

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$\therefore F_c[e^{-x^2}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_{-\infty}^\infty e^{-x^2}\cos sx\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2}\cos sx\,dx$$

$$= \text{R.P of}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2}e^{isx}\,dx$$

$$= \text{R.P of}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2+isx}\,dx$$

$$= \text{R.P of}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2+isx}\;e^{s^2/4}\,e^{-s^2/4}dx$$

$$= \text{R.P of}\frac{e^{-s^2/4}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\left(-x^2+isx+s^2/4\right)}\,dx$$

$$= \text{R.P of}\frac{e^{-s^2/4}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(x-\frac{is}{2}\right)^2}\,dx$$

Put $x - \frac{is}{2} = y, dy = dx$

$$F_c[e^{-x^2}] = \text{R.P of}\frac{e^{-s^2/4}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-y^2}\,dy$$

$$F_c[e^{-x^2}] = \text{R.P of}\frac{e^{-s^2/4}}{\sqrt{2\pi}}\sqrt{\pi}$$

$$\left[\because \int_{-\infty}^\infty e^{-y^2}\,dy = \sqrt{\pi}\right]$$

$$F_c[e^{-x^2}] = \frac{e^{-s^2/4}}{\sqrt{2}}$$

UNIT – V Z - TRANSFORMS AND DIFFERENCE EQUATIONS

PART – A

Problem 1 What is the Z- transform of discrete unit step function?

Solution: Discrete Unit step function is **[CO 5 - H2 -May/Jun 2014]**

$$u(n) = 1,\ n \geq 0$$
$$= 0,\ n < 0$$

$$Z\left[u(n)\right] = \sum_{n=0}^{\infty} 1 z^{-n} = 1 + z^{-1} + z^{-2} + .. = \frac{1}{1-z^{-1}} \text{ if } \left|z^{-1}\right| < 1$$
$$\frac{z}{z-1} \text{ if } |z| > 1$$

Problem 2 Find $Z\left(\frac{1}{2^n}\right)$. **[CO 5 - H2]**

Solution: $Z\left(\frac{1}{2^n}\right) = \sum_{0}^{\infty} \frac{1}{2^n} z^{-n} = 1 + \frac{1}{2} z^{-1} + \frac{1}{2^2} z^{-2} + \frac{1}{2^3} z^{-3} + \ldots$

$$= 1 + \frac{1}{2z} + \frac{1}{4z^2} + \frac{1}{8z^3} + \ldots$$

Problem 3 Find $Z\left[u(n-1)\right]$. **[CO 5 - H2]**

Solution:

$$Z\left[u(n-1)\right] = \sum_{n=0}^{\infty} u(n-1) z^{-n} = \sum_{n=1}^{\infty} u(n-1) z^{-n}$$
$$= \sum_{n=1}^{\infty} z^{-n} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots.$$
$$= \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots.\right]$$
$$= \frac{1}{z}\left(1 - \frac{1}{z}\right) = \frac{1}{z-1} \text{ if } |z| > 1$$

Problem 4 Find the Z- transform of unit impulse function? **[CO 5 - H2 - Nov/Dec 2014]**

Solution: Unit impulse function is

$$\delta(n) = 1, \quad n = 0$$
$$= 0, \quad n \neq 0$$

$$Z\left[\delta(n)\right] = \sum_{n=0}^{\infty} \delta(n) z^{-n} = z^{-0} = 1.$$

Problem 5 Find $Z\left[\delta(n-k)\right]$. **[CO 5 - H2]**

Solution: $Z\left[\delta(n-k)\right]=\sum_{n=0}^{\infty}\delta(n-k)z^{-n}=z^{-k}=\frac{1}{z^{k}}$.

Problem 6 If $Z\left[f(n)\right]=U(z)$ then $Z\left[a^{n}f(n)\right]=$____________ **[CO 5 - H2 -May/Jun 2015]**

Solution: $Z\left[a^{n}f(n)\right]=\sum a^{n}f(n)z^{-n}=\sum f(n)(a/z)^{n}=U(z/a)$.

Problem 7 If $Z\left[f(n)\right]=U(z)$, then show that $Z\left[f(n+k)\right]=z^{\pm k}U(z)$. **[CO 5 - H2]**

Solution:

$$Z\left[f(n\pm k)\right]=\sum_{0}^{\infty}f(n\pm k)z^{-n}=z^{\pm k}\sum_{0}^{\infty}f(n\pm k)z^{-(n\pm k)}$$

$$=z^{\pm k}\sum_{0}^{\infty}f(r)z^{-r}=z^{\pm k}U(z)$$

Problem 8 IF $z\left[f(n)\right]=U(z)$, then $Z\left(\frac{f(n)}{n}\right)=-\int z^{-1}U(z)dz$. **[CO 5 - H2]**

Solution:

$$Z\left[\frac{f(n)}{n}\right]=\sum_{0}^{\infty}\frac{f(n)}{n}z^{-n}=-\sum_{0}^{\infty}f(n)\int z^{-n-1}dz,\qquad \text{since } \frac{z^{-n}}{n}=-\int z^{-n-1}dz$$

$$=-\sum_{0}^{\infty}f(n)\int z^{-n-1}dz=-\int\left(z^{-1}\sum f(n)z^{-n}\right)dz$$

$$=-\int z^{-1}\,U(z)dz.$$

Problem 9 If $Z\left[f(n)\right]=U(z)$, then $Z\left[n\,f(n)\right]=$____________. **[CO 5 - H2 -May/Jun 2013]**

Solution:

$$Z\left[nf(n)\right]=\sum nf(n)z^{-n}=-z\sum -nf(n)z^{-n-1}$$

$$=-\,-z\sum f(n)\frac{d}{dz}\left(z^{-n}\right)=-z\frac{d}{dz}\sum f(n)z^{-n}=-z\frac{d}{dz}U(z).$$

Problem 10 State initial and final value theorem of Z - transform. **[CO 5 - H2 - Nov/Dec 2015]**

Solution: Initial value Theorem

$$If\ \ Z\left[f(n)\right]=U(z), n\geq 0 \text{ then } \underset{n\to 0}{Limit}\, f(n)=f(0)=\underset{z\to\infty}{Limit}\, U(z)$$

Final value Theorem

$$If\ Z\left[f(n)\right]=U(z), n\geq 0\ \ \underset{n\to\infty}{Limit}\, f(n)=\underset{z\to 1}{Limit}(z-1)U(z)$$

Problem 11 Define convolution of two sequences $\{f(n)\}$ and $\{g(n)\}$ **[CO 5 - H2 -May/Jun 2015]**

Solution: $f(n)*g(n)=\sum_{m=0}^{n}f(m)g(n-m)$

Problem 12 Find $Z\left[a^{n+3}\right]$. **[CO 5 - H2]**

Solution: $Z\left[a^{n+3}\right]=\sum a^{n+3}z^{-n}=a^3Z\left(a^n\right)\ =a^3Z\left[a^{n}.1\right]=a^3\dfrac{z/a}{z/a^{-1}}=a^3\dfrac{z}{z-a}$

Problem 13 $Z\left[y_{n+2}\right]=$______________. **[CO 5 - H2]**

Solution: $Z\left[y_{n+2}\right]=z^2\left[y(z)-y_0-y_1z^{-1}\right]$

Problem 14 Find $Z^{-1}\left[\dfrac{z}{(z-1)(z-2)}\right]$. **[CO 5 - H2 -May/Jun 2013]**

Solution: Let $\left[\dfrac{z}{(z-1)(z-2)}\right]=\ U(z)$

$$\text{Then } \frac{U(z)}{z}=\frac{1}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$$

$$z=2,\quad 1=B$$

$$z=1,\quad 1=-A$$

$$\therefore \frac{U(z)}{z}=\frac{-1}{(z-1)}+\frac{1}{z-2}$$

$$\therefore U(z)=\frac{-z}{z-1}+\frac{z}{z-2}$$

$$\therefore u(n)=-1+2^n=2^n-1$$

Problem 15 Write the formula to find the inverse Z- Transform using residue method.

Solution: The inverse Z-Transform of $U(z)$ is given by the formula **[CO 5 - H2]**

$$u_n=\frac{1}{2\pi i}\int_C U(z)z^{n-1}dz$$

= Sum of residues of $U(z)z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the region of convergence given.

PART – B

1. Using Z-Transforms, solve $y_{n+2}+4y_{n+1}-5y_n=24n-8$ **given that** $y_0=3 \ and \ y_1=-5$
[CO 5 - H2 -Nov/Dec 2014]

$$sol: \quad Given \quad y_{n+2}+4y_{n+1}-5y_n=24n-8$$

$$Z[y_{n+2}]+4Z\left[y_{n+1}\right]-5Z[y_n]=24Z[n]-8Z[1]$$

$$[z^2Y(z)-z^2y(0)-zy(1)]+4[zY(z)-zy(0)]-5Y(z)=\frac{24\,z}{(z-1)^2}-\frac{8z}{z-1}$$

$$put \quad y_0=3, y_1=-5$$

$$\therefore\ [z^2+4z-5]Y(z)-3z^2+5z-12z=\frac{24\,z-8z(z-1)}{(z-1)^2}$$

$$(z+5)(z-1)Y(z) = \frac{24\,z-8z^2+8z}{(z-1)^2}+3z^2+7z$$

$$(z+5)(z-1)Y(z) = \frac{3z^4+z^3-19z^2+39z}{(z-1)^2}$$

$$Y(z) = \frac{z[3z^3+z^2-19z^1+39]}{(z+5)(z-1)^3}$$

$$\frac{Y(z)}{z} = \frac{[3z^3+z^2-19z^1+39]}{(z+5)(z-1)^3}$$

$$let\ \frac{[3z^3+z^2-19z^1+39]}{(z+5)(z-1)^3}=\frac{A}{z+5}+\frac{B}{z-1}+\frac{C}{(z-1)^2}+\frac{D}{(z-1)^3}$$

$$3z^3+z^2-19z^1+39 = A(z-1)^3+B(z+5)(z-1)^2+C(z+5)(z-1)+D(z+5)$$

$$solving \ \ this\ eqn, we\ get\ A=1, B=2, C=-2, D=4$$

$$\frac{Y(z)}{z}=\frac{1}{z+5}+\frac{2}{z-1}+\frac{-2}{(z-1)^2}+\frac{4}{(z-1)^3}$$

$$Y(z)=\frac{z}{z+5}+\frac{2z}{z-1}-\frac{2z}{(z-1)^2}+\frac{4z}{(z-1)^3}$$

$$y(n)=Z^{-1}\left[\frac{z}{z+5}\right]+2Z^{-1}\left[\frac{z}{z-1}\right]-2Z^{-1}\left[\frac{z}{(z-1)^2}\right]+2Z^{-1}\left[\frac{2z}{(z-1)^3}\right]$$

$$y(n)=(-5)^n+2(1)^n-2n+2n(n-1)$$

2. Using Z-Transforms, solve $y_{n+2}+6y_{n+1}+9y_n=2^n$ given that $y_0=0 \ and \ y_1=0$
[CO 5 - H2 - Nov/Dec 2012]

$$sol: \quad Given \quad y_{n+2}+6y_{n+1}+9y_n=2^n$$

$$Z[y_{n+2}]+6Z\left[y_{n+1}\right]+9Z[y_n]=Z[2^n]$$

$$[z^2Y(z)-z^2y(0)-zy(1)]+6[zY(z)-zy(0)]+9Y(z)=\frac{z}{z-2}$$

$put \quad y_0 = 0, y_1 = 0$

$$\therefore \; [z^2 + 6z + 9]Y(z) = \frac{z}{z-2}$$

$$(z+3)^2 Y(z) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} \quad ----------(1)$$

$$let \; \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z+3)(z-2) + C(z-2)$$

$$solving \; this \; eqn, we \; get \; A = \frac{1}{25}, B = \frac{-1}{25}, C = \frac{-1}{5}$$

$$\therefore Eqn \; (1) \; becomes \quad \frac{Y(z)}{z} = \left[\frac{1}{25}\right]\frac{1}{z-2} - \left[\frac{1}{25}\right]\frac{1}{z+3} - \left[\frac{1}{5}\right]\frac{1}{(z-3)^2}$$

$$Y(z) = \frac{1}{25}\left[\frac{z}{z-2}\right] - \frac{1}{25}\left[\frac{z}{z+3}\right] - \frac{1}{5}\left[\frac{z}{(z+3)^2}\right]$$

$$y(n) = \frac{1}{25}Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5}Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$y(n) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{n}{5}(-3)^n$$

3. Find z $\left[\frac{1}{n+1}\right]$. [CO 5 - H2]

Solution:

z [f (n)] = $\sum_{n=0}^{\infty} f\,(n).z^{-n}$

z $\left[\frac{1}{n+1}\right]$ = $\sum_{n=0}^{\infty} \frac{1}{n+1}.z^{-n}$

= 1 + $\frac{1}{2}$. z^{-1} + $\frac{1}{3}$. z^{-2} + $\frac{1}{4}$. z^{-3} + …….

= 1 + $\frac{1/2}{z}$ + $\frac{1/3}{z^2}$ + …….

= z $\left[\frac{1}{z} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \;....\right]$

= z $\left[-\log\left(1 - \frac{1}{z}\right)\right]$

= -z log $\left(1 - \frac{1}{z}\right)$

= -z log $\left(\frac{z-1}{z}\right)$

= z log $\left(\frac{z}{z-1}\right)$

(ii). Find $z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{4}\right)^2}\right]$ using convolution theorem. **[CO 5 - H2]**

Solution.

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=Z^{-1}\left[\frac{z}{(z-a)}\cdot\frac{z}{(z-a)}\right]$$

$$=Z^{-1}\left[\frac{z}{(z-a)}\right]*Z^{-1}\left[\frac{z}{(z-a)}\right]$$

$$=a^n*a^n$$

$$=\sum_{r=0}^{n}a^r a^{n-r}$$

$$=a^n\sum_{r=0}^{n}a^r a^{-r}$$

$$=a^n\sum_{r=0}^{n}1$$

$$=a^n(n+1)$$

$$=(n+1)a^n$$

Therefore

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{4}\right)^2}\right]=(n+1)\left(\frac{1}{4}\right)^n$$

4. Solve $y_{n+3}-3y_{n+1}+2y_n=0$ *given that* $y_0=4;\ y_1=0;\ y_2=8$. **[CO 5 - H2 - Apr/May 2014]**

Given that

$$y_{n+3}-3y_{n+1}+2y_n=0$$

$$Z[y_{n+3}]-3Z[y_{n+1}]+2Z[y_n]=Z[0]$$

$$z^3Z[y_n]-z^3y_0-z^2y_1-zy_2$$

$$-3\{zZ[y_n]-zy_0\}+2Z[y_n]=0$$

given that $y_0=4;\ y_1=0;\ y_2=8$

$\therefore$ $$z^3Z[y_n]-z^3(4)-0-z(8)$$

$$-3\{zZ[y_n]-z(4)\}+2Z[y_n]=0$$

$$z^3Z[y_n]-3zZ[y_n]+2Z[y_n]-4z^3-8z+12z=0$$

$\left(z^3-3z+2\right)Z\left[y_n\right]=4z^3-4z$

$$Z\left[y_n\right]=\frac{4z\left(z^2-1\right)}{\left(z^3-3z+2\right)}$$

$$Z\left[y_n\right]=\frac{4z\left(z-1\right)\left(z+1\right)}{\left(z-1\right)\left(z+2\right)\left(z-1\right)}$$

$$Z\left[y_n\right]=\frac{4z\left(z+1\right)}{\left(z+2\right)\left(z-1\right)}$$

$$\frac{Z\left[y_n\right]}{z}=\frac{4\left(z+1\right)}{\left(z+2\right)\left(z-1\right)} \qquad -------(1)$$

Consider

$$\frac{4\left(z+1\right)}{\left(z+2\right)\left(z-1\right)}=\frac{A}{\left(z+2\right)}+\frac{B}{\left(z-1\right)}$$

$$4\left(z+1\right)=A\left(z-1\right)+B\left(z+2\right)$$

Put $z=-2$

$$4\left(-1\right)=A\left(-3\right)+0$$

$$\frac{4}{3}=A$$

Put $z=1$

$$4\left(2\right)=0+B\left(3\right)$$

$$\frac{8}{3}=B$$

Hence,

$$\frac{4\left(z+1\right)}{\left(z+2\right)\left(z-1\right)}=\frac{\left(\frac{4}{3}\right)}{\left(z+2\right)}+\frac{\left(\frac{8}{3}\right)}{\left(z-1\right)}$$

Putting in equation(1)*, we get*

$$\frac{Z\left[y_n\right]}{z}=\frac{\left(\frac{4}{3}\right)}{\left(z+2\right)}+\frac{\left(\frac{8}{3}\right)}{\left(z-1\right)}$$

$$Z\left[y_n\right]=\left(\frac{4}{3}\right)\frac{z}{\left(z+2\right)}+\left(\frac{8}{3}\right)\frac{z}{\left(z-1\right)}$$

$$\left[y_n\right]=\left(\frac{4}{3}\right)Z^{-1}\left[\frac{z}{\left(z+2\right)}\right]+\left(\frac{8}{3}\right)Z^{-1}\left[\frac{z}{\left(z-1\right)}\right]$$

$$y_n=\left(\frac{4}{3}\right)(-2)^n+\left(\frac{8}{3}\right)$$

5. Show that $Z\left[f\left(n+1\right)\right]=zF\left(z\right)-zf\left(0\right)$ *if* $Z\left[f\left(n\right)\right]=F\left(z\right)$. **[CO 5 - H2 - Apr/May 2014]**

$$We\ have, \qquad Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z\left[f(n+1)\right] = \sum_{n=0}^{\infty} f(n+1) z^{-n}$$

$$Put\ \ n+1 = m \quad \Rightarrow n = m-1$$

and

n	0	∞
m	1	∞

Therefore,

$$Z\left[f(n+1)\right] = \sum_{m=1}^{\infty} f(m) z^{-n+1}$$

$$= z\sum_{m=1}^{\infty} f(m) z^{-n}$$

$$= z\left\{\sum_{m=0}^{\infty} f(m) z^{-n} - f(0)\right\}$$

$$= zF(z) - zf(0)$$

6. State and prove the initial value theorem. **[CO 5 - H2 - Nov/Dec 2015]**

State:

$$Z\left[f(t)\right] = F(z),\ then\ \lim_{t\to 0} f(t) = \lim_{z\to\infty} F(z)$$

Proof :

$$F(z) = Z\left[f(t)\right]$$

$$= \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \left(\frac{1}{z}\right)^{n}$$

$$= f(0) + f(T)\left(\frac{1}{z}\right) + f(2T)\left(\frac{1}{z}\right)^{2} + \ldots\ldots\ldots\ldots\ldots$$

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left\{f(0) + f(T)\left(\frac{1}{z}\right) + f(2T)\left(\frac{1}{z}\right)^{2} + \ldots\ldots\ldots\ldots\ldots\right\}$$

$$= f(0) + 0 + 0 + \ldots\ldots\ldots$$

$$= f(0)$$

$$= \lim_{t\to 0} f(t)$$

Hence the proof

7. State and prove the final value theorem **[CO 5 - H2 - Nov/Dec 2015]**

State :–

$$If\ \ Z\left[f(t)\right]=F(z),\ then\ \lim_{t\to\infty}f(t)=\lim_{z\to1}(z-1)F(z)$$

Pr*rof* :–

$$we\ \ have\ \ Z\left[f(t+T)\right]-Z\left[f(t)\right]=\sum_{n=0}^{\infty}f(nT+T)z^{-n}-\sum_{n=0}^{\infty}f(nT)z^{-n}$$

$$zF(z)-zf(0)-F(z)=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]z^{-n}$$

$$(z-1)F(z)-zf(0)=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]z^{-n}$$

$$\lim_{z\to1}\left\{(z-1)F(z)-zf(0)\right\}=\lim_{z\to1}\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]z^{-n}$$

$$\lim_{z\to1}(z-1)F(z)-f(0)\ =\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]$$

$$\begin{aligned}=\lim_{n\to\infty}[\,&f(T)-f(0)\\&+f(2T)-f(T)\\&+f(3T)-f(2T)\\&+\dots\dots\dots\dots\\&+f(nT)-f((n-1)T)]\end{aligned}$$

$$=\lim_{n\to\infty}\left[f(nT)-f(0)\right]$$

$$=f(\infty)-f(0)$$

$$\lim_{z\to1}(z-1)F(z)=f(\infty)$$

$$=\lim_{t\to\infty}f(t)$$

Hence the proof

(ii). Find z $\left[a^{|n|}\right]$. **[CO 5 - H2]**

Solution:

$$z\,[f(n)]=\sum_{n=-\infty}^{\infty}f(n)\,.\,z^{-n}$$

$$\therefore\ z\left[a^{|n|}\right]=\sum_{n=-\infty}^{\infty}a^{|n|}\,.\,z^{-n}$$

$$=\sum_{n=-\infty}^{-1}a^{-n}\,.\,z^{-n}+\sum_{n=0}^{\infty}a^{n}\,.\,z^{-n}\qquad\left[\because\ |n|=\begin{cases}-n, & (-\infty,\ \ -1)\\ n, & (0,\ \ \infty)\end{cases}\right]$$

$$=\sum_{n=1}^{\infty}a^{n}\,.\,z^{n}+\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{n}$$

$= [az + (az)^2 + (az)^3 + \ldots..] + [1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \ldots\ldots.]$

$= az\,[1 + az + (az)^2 + \ldots..] + [1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \ldots\ldots.]$

$= az\,(1 - az)^{-1} + \left(1 - \frac{a}{z}\right)^{-1}$ $\{\because 1 + x + x^2 + \ldots\ldots = (1 - x)^{-1}\}$

$= \frac{az}{1-az} + \frac{z}{z-a}$

$= \frac{az\,(z-a) + z\,(1-az)}{(1-az)(z-a)}$

$= \frac{z\,(1-a^2)}{(1-az)(z-a)}$

8. Find Z^{-1} $\left[\frac{Z^2-3Z}{(Z-5)(Z+2)}\right]$. [CO 5 - H2]

Let F (Z) = $\frac{Z^2-3Z}{(Z-5)(Z+2)}$

$\frac{F(Z)}{Z} = \frac{Z-3}{(Z-5)(Z+2}$

$\frac{Z-3}{(Z-5)(Z+2} = \frac{A}{Z-5} + \frac{B}{Z+2} \longrightarrow$ (1)

Z-3 = A (Z+2) + B (Z-5)

Put Z= 5, Put Z = -2

7A = 2 -7B= -5

$\Rightarrow$ A= $\frac{2}{7}$ $\Rightarrow$ B= $\frac{5}{7}$

$\frac{F(Z)}{Z} = \frac{2}{7}\,\frac{1}{Z-5} + \frac{5}{7} + \frac{1}{Z+2}$

$\frac{F(Z)}{Z} = \frac{2}{7}\,\frac{Z}{Z-5} + \frac{5}{7} + \frac{Z}{Z+2}$

$Z^{-1}\,[F(Z)] = \frac{2}{7}\,Z^{-1}\left[\frac{Z}{Z-5}\right] + \frac{5}{7}\,Z^{-1}\left[\frac{Z}{Z+2}\right]$

$Z^{-1}\left[\frac{Z^2-3Z}{(Z-5)(Z+2)}\right] = \frac{2}{7}\,5^n + \frac{5}{7}\,(-2)^n, n \geq 0$

9. Find $Z^{-1}\left[\frac{Z^2}{Z^2+4}\right]$. **[CO 5 - H2]**

Let F(z) = $\frac{Z^2}{Z^2+4}$, Z^{-1} $[F(Z)]$ = f (n)

Z^{n-1} F(z) = $\frac{Z^{n+1}}{Z^2+4}$

The poles are Z = $2i$, Z = - $2i$ (simple poles)

Res $\{Z^{n-1}\,F(Z)\}_{Z=2i}$ = Z→$2i$, $(\cancel{Z - 2i})\ \frac{Z^{n+1}}{(Z+2i)(\cancel{Z-2i})}$

$$= \frac{(2i)^{n+1}}{4i} = \frac{2i^n}{2} = 2^{n-1}\,(i)^n$$

Res $\{Z^{n-1}\,F(Z)\}_{Z=2i}$ = Z→$-2i$, $(\cancel{Z+2i})\ \frac{Z^{n+1}}{(\cancel{Z+2i})(Z-2i)}$

$$= \frac{(-2i)^{n+1}}{-4i} = \frac{(-2i)(-2i)^n}{-4i}$$

$$= \frac{1}{2}\,2^n\,(-i)^n = 2^{n-1}\,(-i)^n$$

$f\,(n)$ sum of the residue of Z^{n-1} F(Z) at its poles

$$= 2^{n-1}\,i^n + 2^{n-1}\,(-i)^n$$

$$= 2^{n-1}\left[\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]$$

$$= 2^n\cos\frac{n\pi}{2} \qquad \left[\therefore (i)^n = \cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right]$$

$$f\,(n) = 2^n\cos\frac{n\pi}{2} \qquad \left[\therefore (-i)^n = \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]$$